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SOLVABLE LIE ALGEBRAS OF DIMENTION FIVE

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1. Introduction. In [3] we developed an inductive method for classification of solvable Lie algebras. More precisely, let $\mathcal{G}$ and $\mathfrak{a}$ be solvable Lie algebras over any commutative field F, $\mathfrak{a}$ abelian, and assume that $\mathfrak{n}$ is a fixed nilpotent ideal of $\mathcal{G}$ containing the commutator subalgebra $[\mathcal{G},\mathcal{G}]$. If $\theta : \mathcal{G} \to \operatorname{End} \mathfrak{a}$ is a Lie representation we may consider the second cohomolgy space $H^2(\mathcal{G},\theta)$ of $\mathcal{G}$ with coefficients in $\mathfrak{a}$, which we identify to a certain quotient of the linear space of all alternating bilinear functions $B:\mathcal{G}\times\mathcal{G} \to \mathfrak{a}$ satisfying the cocycle identity,

$$B(X,[Y,Z])+B(Z,[X,Y])+B(Y,[Z,X])+\theta(X)B(Y,Z)+\theta(Z)B(X,Y)+\theta(Y)B(Z,X)=0,$$

for all X, Y, Z in $\mathcal{G}$; factoring out the space of all exact bilinear functions of the form

$$df(X,Y) = \theta(X)f(Y)-\theta(Y)f(X)-f[X,Y], \quad \text{where } f:\mathcal{G}\to\mathfrak{a} \text{ is linear.}$$

Let $H^2(\mathcal{G},\mathcal{G}/\mathfrak{n},\mathfrak{a}) = \bigcup\limits_{\theta} H^2(\mathcal{G},\theta)$ where the union runs over the family of all Lie representations θ of $\mathcal{G}$ in $\mathfrak{a}$ satisfying

(1.1) $\ker\theta \supset \mathfrak{n}$,

(1.2) For all X in the nilradial of $\mathcal{G}$, $\theta(X)$ is nilpotent if and only if X is in $\mathfrak{n}$.

There is a natural action of $\operatorname{Aut}\mathcal{G} \times \operatorname{Aut}\mathfrak{a}$ in $\bigcup\limits_{\theta} H^2(\mathcal{G},\theta)$ (θ arbitrary), given by

$$(\alpha,\psi,B) \to \psi\circ B\circ\alpha \ ; \ \alpha\in\operatorname{Aut}\mathcal{G},\ \psi\in\operatorname{Aut}\mathfrak{a},\ B\in\bigcup_{\theta}H^2(\mathcal{G},\theta).$$

If $B\in H^2(\mathcal{G},\theta)$, let $B^0 = B|_{\mathfrak{n}\times\mathfrak{n}}$, and $\mathfrak{z}_{B^0} = \{X\in\mathfrak{n}: B(X,\mathfrak{n})=(0)\}$, and put $\mathfrak{z}(\theta) = \{a\in\mathfrak{a}: \theta(\mathcal{G})a = (0)\}$.

<u>1.1. Definition</u>. Let $\mathcal{G}$ and $\tilde{\mathcal{G}}$ be solvable Lie algebras, and assume $\tilde{\mathcal{G}}$ is an extension of $\mathcal{G}$ by an abelian Lie algebra $\mathfrak{a}$. Let $\tilde{\mathfrak{n}}$ be the nilradical of $\tilde{\mathcal{G}}$, and suppose $\mathfrak{n}$ is a nilpotent ideal of $\mathcal{G}$ containing the commutator subalgebra $[\mathcal{G},\mathcal{G}]$. We say that the extension $\tilde{\mathcal{G}}$ is <u>$\mathfrak{n}$-admissible</u> if the following conditions hold

(1) $\tilde{\mathcal{G}}$ contains no non-zero abelian direct factor.

(2) $\tilde{\mathfrak{n}}/\mathfrak{a} \approx \mathfrak{n}$

(3) $\mathfrak{a}$ is the center of $\tilde{\mathfrak{n}}$

In case $[\mathcal{G},\mathcal{G}]$ is equal to the nilradical of $\mathcal{G}$, we simply say that such an extension $\tilde{\mathcal{G}}$ is <u>admissible</u>.

We are now ready to restate the main result of [3].

<u>Theorem 1</u>. Let $\mathcal{G}$ and $\mathfrak{a}$ be solvable Lie algebras over any commutative field F, $\mathfrak{n}$ a nilpotent ideal of $\mathcal{G}$ containing $[\mathcal{G},\mathcal{G}]$. The isomorphism classes of all $\mathfrak{n}$-admissible extensions $\tilde{\mathcal{G}}$ of $\mathcal{G}$ by $\mathfrak{a}$ are in bijective correspondence with the family of all $\operatorname{Aut}\mathcal{G} \times \operatorname{Aut}\mathfrak{a}$ orbits Ω in $\bigcup_\theta H^2(\mathcal{G},\theta)$ which satisfy the following conditions,

(1) $\Omega \cap H^2(\mathcal{G},\mathcal{G}/\mathfrak{n},\mathfrak{a}) \neq \emptyset$

(2) If $B \in \Omega \cap H^2(\mathcal{G},\theta)$ and $\ker\theta \supset \mathfrak{n}$, then $\mathfrak{a}$ may not be written as a direct sum $\mathfrak{a} = \mathfrak{B} \oplus \mathfrak{D}$ where $\mathfrak{B}$ is θ-invariant and contains the range space $B(\mathcal{G},\mathcal{G})$ of B, and where $(0) \neq \mathfrak{D} \subset \mathfrak{F}(\theta)$.

(3) $\mathfrak{F}_{B^0} \cap \mathfrak{z} = (0)$, where $\mathfrak{z}$ is the center of $\mathfrak{n}$.

We shall comment briefly on the various hypothesis of the above theorem. (1.2) means that $\tilde{\mathfrak{n}}/\mathfrak{a} \approx \mathfrak{n}$, and (1.1) ensures that $\mathfrak{a}$ is central in $\tilde{\mathfrak{n}}$. Thus $\tilde{\mathfrak{n}}$ becomes a central extension of $\mathfrak{n}$ by $\mathfrak{a}$.

Furthermore, $\mathcal{J}_{B^0} \cap \mathfrak{z} = (0)$ if and only if the center of $\tilde{\mathfrak{n}}$ is $\mathfrak{a}$. Finally, condition (2) ensures that the extension $\tilde{\mathfrak{g}}$ contains no abelian direct factors (except (0) and $\tilde{\mathfrak{g}}$).

The purpose of the present article is to present, on the basis of the above theorem, a list of all real solvable Lie algebras of dimension five, containing no nontrivial direct factors. To complete this program it will be convenient to reformulate Theorem 1 as follows.

Let $\mathfrak{g}$, $\mathfrak{a}$, and $\mathfrak{n}$ be as above. If $\theta_i : \mathfrak{g} \to \operatorname{End}\mathfrak{a}$ is a representation, $i = 1,2$, we say that θ_1 is <u>equivalent</u> with θ_2 in case there exists an invertible linear operator ψ on $\mathfrak{a}$ such that $\psi\,\theta_1\,\psi^{-1} = \theta_2$. Let $\hat{\mathfrak{g}}(\mathfrak{n},\mathfrak{a})$ be the set of all equivalence classes of Lie representations θ of $\mathfrak{g}$ in $\mathfrak{a}$ such that $\ker\theta \supseteq \mathfrak{n}$. In order to construct all extensions $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$ by $\mathfrak{a}$ satisfying the hypothesis of Theorem 1 it suffices to pick exactly one representative θ from each equivalence class in $\hat{\mathfrak{g}}(\mathfrak{n},\mathfrak{a})$ and then to consider the orbit spaces $H^2(\mathfrak{g},\theta)/K(\theta)$ where
$K(\theta) = \{(\alpha,\psi) \in \operatorname{Aut}\mathfrak{g} \times \operatorname{Aut}\mathfrak{a} : \ \psi(\theta\circ\alpha)(\cdot)\psi^{-1} = \theta\}$ is the fixed point group of θ in $\operatorname{Aut}\mathfrak{g} \times \operatorname{Aut}\mathfrak{a}$. For if Ω $H^2(\mathfrak{g},\theta)/K(\theta)$, and $B \in \Omega$, then $\Omega = \{\psi\circ B\circ\alpha : \ (\alpha,\psi) \in \operatorname{Aut}\mathfrak{g} \times \operatorname{Aut}\mathfrak{a}\} \cap H^2(\mathfrak{g},\theta)$. The converse also holds.

Let $\operatorname{Ext}(\mathfrak{g},\mathfrak{a},\mathfrak{n})$ be the family of isomorphism classes of all solvable Lie algebras $\tilde{\mathfrak{g}}$ with nilradical $\tilde{\mathfrak{n}}$ such that $\tilde{\mathfrak{n}}/\mathfrak{a} \approx \mathfrak{n}$, $\tilde{\mathfrak{g}}/\mathfrak{a} \approx \mathfrak{g}$, and $\mathfrak{a}$ = the center of $\tilde{\mathfrak{n}}$. In other words, $\operatorname{Ext}(\mathfrak{g},\mathfrak{a},\mathfrak{n})$ is the set of all isomorphism classes of $\mathfrak{n}$-admissible extensions of $\mathfrak{g}$ by $\mathfrak{a}$, (Definition 1.1).

Let us choose exactly one representation θ from each isomorphism class in $\hat{\mathfrak{g}}(\mathfrak{n},\mathfrak{a})$, and denote the resulting set of representations by $\operatorname{Rep}(\mathfrak{g},\mathfrak{a},\mathfrak{n})$. We can now state our reformulation of Theorem 1.

Theorem 2. With notation as above, $\mathrm{Ext}(\mathcal{G},\mathcal{A},\mathcal{N})$ is in bijective correspondence with those orbits Ω in $\bigcup_{\theta\in \mathrm{Rep}(\mathcal{G},\mathcal{A},\mathcal{N})} H^2(\mathcal{G},\theta)/K(\theta)$ which satisfy the following conditions.

(1) If $B\in\Omega\in H^2(\mathcal{G},\theta)/K(\theta)$ then $\mathcal{A}$ may not be written as a direct sum of subspaces $\mathcal{B}$ and $\mathcal{D}$ where $\mathcal{B}\supset B(\mathcal{G},\mathcal{G})$, $\theta(\mathcal{G})\mathcal{B}\subset\mathcal{B}$, and $(0)\neq\mathcal{D}\subset\mathcal{S}(\theta)$ $(\mathcal{S}(\theta)=\{a\in\mathcal{A}: \theta(\mathcal{G})a=(0)\})$.

(2) $\mathcal{S}_{B^o}\cap\mathcal{Z}=(0)$, $\mathcal{Z}$ = the center of $\mathcal{N}$, $B^o = B|\mathcal{N}\times\mathcal{N}$.

Notation. In the sequel the nilpotent Lie algebras of dimension five without direct factors will be denoted by $\mathcal{N}_{5,1},\ldots,\mathcal{N}_{5,6}$, whereas the solvable, non-nilpotent ones are denoted by $\mathcal{G}_{5,1}$, $\mathcal{G}_{5,2}$, etc.

If $(e_i)_{i=1}^n$ is a (fixed) basis of the Lie algebra $\mathcal{G}$, we define the alternating bilinear forms B_{ij} by $B_{ij}(\sum_{k=1}^n x_k e_k, \sum_{k=1}^n y_k e_k) = x_i y_j - y_i x_j$, $1\le i<j\le n$, $x_k, y_k\in F$. We refer to [1] for a complete list of all real solvable Lie algebras of dimension less than or equal to four.

Now let $\mathcal{G}^*$ be the space of all real linear functionals of $\mathcal{G}$. We shall always work with the basis $\mathcal{E}^* = (e_i^*)_{i=1}^n$ of $\mathcal{G}^*$ dual to the basis $\mathcal{E}$ of $\mathcal{G}$.

If $v_1,v_2,\ldots,v_k$ are elements of a vector space V we let $(v_1,v_2,\ldots,v_k)$ denote the linear subspace of V generated by these vectors.

Let $B:\mathcal{G}\times\mathcal{G}\to\mathcal{A}$ be a bilinear map, then we always write $\Sigma B(X,[Y,Z])$ for the sum $B(X,[Y,Z])+B(Z,[X,Y])+B(Y,[Z,X])$; $X,Y,Z\in\mathcal{G}$. Similarly, if $\theta:\mathcal{G}\to\mathrm{End}\,\mathcal{A}$ is a Lie representation we write $\Sigma\theta(X)B(Y,Z)$, meaning the sum $\theta(X)B(Y,Z)+\theta(Z)B(X,Y)+\theta(Y)B(Z,X)$.

2. Extensions of four dimensional solvable Lie algebras

2.1. Extensions of $\mathcal{G}_{4,1}$. $\mathcal{G} = \mathcal{G}_{4,1}$ is defined by the relations $[e_1,e_3] = e_3$, $[e_1,e_4] = e_4$, $[e_2,e_3] = e_4$ between the elements of a basis $\mathcal{E} = (e_i)_{i=1}^{4}$.

Let $\mathfrak{a} = \mathcal{G}_1 = Re_5$. The extension of $\mathcal{G}$ by $\mathfrak{a}$ are determined by a skew symmetric bilinear form $B : \mathcal{G} \times \mathcal{G} \to \mathcal{G}_1$ together with a representation $\theta : \mathcal{G} \to \operatorname{End} \mathcal{G}_1$. In this case $[\mathcal{G},\mathcal{G}] = Re_3 \oplus Re_4$, and the nilradical $\mathfrak{n}_0$ of $\mathcal{G}$ is $Re_2 \oplus Re_3 \oplus Re_4 \quad \mathcal{G}_3$, the Heisenberg algebra. Hence the only possible choices of nilpotent subalgebra $\mathfrak{n}$ (as in Theorem 1) are $\mathfrak{n} = \mathfrak{n}_0$ and $\mathfrak{n} = [\mathcal{G},\mathcal{G}]$. But if $\mathfrak{n} = [\mathcal{G},\mathcal{G}]$ $\theta(e_2)$ must be nilpotent since $e_2 \in \mathfrak{n}_0$ (Thm.1). Now $\mathfrak{a} = \mathcal{G}_1$ so that $\theta(e_2) = 0$. Hence we are left with the case

$$\mathfrak{n} = \mathfrak{n}_0 , \quad \theta(e_1)e_5 = \tilde{\theta}e_5 , \quad \tilde{\theta} \in R , \quad \ker\theta = \mathfrak{n}_0 .$$

Let $f : \mathcal{G} \to Re_5 = \mathfrak{a}$ be linear. The trivial cocycles of $\mathcal{G}$ (coboundaries) are of the form

$$\begin{aligned} df(X,Y) &= \theta(X)f(Y) - \theta(Y)f(X) - f[X,Y] . \\ &= (\tilde{\theta}f_2B_{12} + (\tilde{\theta}-1)f_3B_{13} + (\tilde{\theta}-1)f_4B_{14} - f_4B_{23})(X,Y) \end{aligned}$$

where $f = (f_1,f_2,f_3,f_4)$ relative to a dual basis of $\mathcal{E}$ and $B_{ij}(X,Y) = x_iy_j - x_jy_i$, $1 \leq i < j \leq 4$.

Hence (identifying θ with $\tilde{\theta}$),

$$B^2(\mathcal{G},0) = (B_{13}, B_{14}+B_{23}) \tag{2.1}$$

$$B^2(\mathcal{G},1) = (B_{12}, B_{23}) \tag{2.2}$$

$$B^2(\mathcal{G},\theta) = (B_{12}, B_{13}, B_{14}-B_{23}) , \quad \tilde{\theta} \neq 0,1 . \tag{2.3}$$

Furthermore, $\mathcal{S}_{B^0} \cap \mathfrak{Z} = (0)$ only for forms B in the linear span of B_{24} and B_{34} (provided these are cocycles).

Now consider B_{24} :

$$\Sigma B_{24}(X,[Y,Z]) = \Sigma(x_2(y_1z_4 - y_4z_1) + x_2(y_2z_3 - y_3z_2))$$

$$= \Sigma x_2(y_1z_4 - y_4z_1)$$

where $X,Y,Z \in \mathfrak{G}$ and $(x_i),(y_i),(z_i)$ are their coordinates relative to $\mathcal{E}$. Here <u>and below</u> the sums are extended over the cyclic permutations of (X,Y,Z). Further,

$$\Sigma\,\theta(X)B_{24}(Y,Z) = \Sigma\,\tilde{\theta}x_1(y_2z_4 - y_4z_2)$$

so that

$$\Sigma(B_{24}(X,[Y,Z]) - \theta(X)B_{24}(Y,Z))$$

$$= \Sigma(1-\tilde{\theta})x_2(y_1z_4 - y_4z_1) = 0 \iff \tilde{\theta} = 1 .$$

Hence B_{24} is a cocycle iff $\tilde{\theta} = 1$. Similarly one sees that $B_{13} \in H^2(\mathfrak{G},1)$.

Next consider B_{34} :

$$\Sigma(B_{34}(X,[Y,Z]) - \theta(X)B_{34}(Y,Z)) = (2-\tilde{\theta})x_3(y_1z_4 - y_4z_1)$$

Hence B_{34} is a cocycle iff $\tilde{\theta} = 2$.

It follows that

$$H^2(\mathfrak{G},1) = (B_{24},B_{13}) \tag{2.4}$$

$$H^2(\mathfrak{G},2) = (B_{34}) \tag{2.5}$$

These are the only cases to be considered. In order to continue we need the following

<u>2.1.1. Lemma</u>. $\mathrm{Aut}(\mathfrak{G}_{4,1})$ is isomorphic to the group of all real

matrices

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ v & 0 & d & 0 \\ \varphi & av & \psi & ad \end{pmatrix} \text{ , where } ad \neq 0 .$$

Proof. $\mathfrak{g}_{4,1}$ is an extension of $\mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3 \approx \mathfrak{g}_2 \oplus \mathbb{R}e_2$ by $\mathbb{R}e_4$, and $\mathrm{Aut}(\mathfrak{g}_2 \oplus \mathbb{R}e_2)$ is isomorphic to the group of all matrices $\begin{pmatrix} 1 & 0 & 0 \\ u & a & 0 \\ v & c & d \end{pmatrix}$, where $ad \neq 0$. Hence the lemma follows from [3, Corollary 2.6].

Next we find the $\mathrm{Aut}\,\mathfrak{g} \times \mathrm{Aut}\,\mathfrak{a}$ orbits in $H^2(\mathfrak{g},1) \cup H^2(\mathfrak{g},2)$. One has

$$A^t B_{24} A = a^2 d B_{24}$$

$$A^t (B_{13}+B_{24}) A = d(a^2 B_{24}+B_{13})$$

$$A^t (B_{13}-B_{24}) A = d(B_{13}-a^2 B_{24})$$

where the computations are carried out modulo coboundaries. Except for the "degenerate" orbit of B_{13} (recall that $\mathfrak{z}_{B^0_{13}} \cap \mathfrak{z} \neq (0)$) there are three $\mathrm{Aut}\,\mathfrak{g} \times \mathrm{Aut}\,\mathfrak{a}$ orbits in $H^2(\mathfrak{g},1)$, and we obtain three nonisomorphic extensions of $\mathfrak{g}_{4,1}$ in this case. They are

$$\mathfrak{g}_{5,3} = \mathfrak{g}(B_{24},1) , \quad \mathfrak{g}_{5,4} = \mathfrak{g}(B_{24}+B_{13},1) , \text{ and } \mathfrak{g}_{5,5} = \mathfrak{g}(B_{24}-B_{13},1) .$$

Finally $H^2(\mathfrak{g},2)$ consists of one single orbit and we find the extension

$$\mathfrak{g}_{5,2} = \mathfrak{g}(B_{34},2) .$$

2.1.2. Proposition. Let $\mathfrak{g} = \mathfrak{g}_{4,1}$, $\mathfrak{a} = \mathfrak{g}_1$. The extensions of $\mathfrak{g}$ by $\mathfrak{a}$, satisfying the hypothesis of Theorem 1 are, within isomorphisms,

$$\mathfrak{g}_{5,2} = \mathfrak{g}(B_{34},2)\ ,\quad \mathfrak{g}_{5,3} = \mathfrak{g}(B_{34},2)$$

$$\mathfrak{g}_{5,4} = \mathfrak{g}(B_{24}+B_{13},1)\ ,\quad \mathfrak{g}_{5,5} = \mathfrak{g}(B_{24}-B_{13},1)\ .$$

These Lie algebras are pairwise non-isomorphic.

2.2 Extensions of $\mathfrak{g}_{4,2}$. Let $\varepsilon = \{e_1,e_2,e_3,e_4\}$ be a basis for the linear space R^4. Put $[e_1,e_3] = e_3$, $[e_1,e_4] = e_4$, $[e_2,e_3] = -e_4$, $[e_2,e_4] = e_3$. This gives the Lie algebra $\mathfrak{g} = \mathfrak{g}_{4,2}$. Its nilradical and commutator subalgebra both equal $Re_3 + Re_4$.
Let $\mathfrak{n}_0 = Re_3 + Re_4$, and let $\theta : \mathfrak{g} \to End(Re_5)$, $\theta(e_i)e_5 = \theta_i e_5$, $i = 1,2$, and $\ker\theta \supset \mathfrak{n}_0$. We shall determine the space of coboundaries $B^2(\mathfrak{g},\theta)$. Thus let $f : \mathfrak{g} \to Re_5$ be linear, $f = (f_i)_{i=1}^4$ relative to ε^*. Then, for $X = (X_i)_{i=1}^4$, $Y = (Y_i)_{i=1}^4$,

$$\begin{aligned} df(X,Y) &= \theta(X)f(Y) - \theta(Y)f(X) - f[X,Y] \\ &= ((\theta_1 f_2-\theta_2 f_1)B_{12} + (\theta_1-1)f_3\, B_{13} + (\theta_2 f_4-f_3)B_{24} + (\theta_1-1)f_4 B_{14} \\ &\quad + (\theta_2 f_3-f_4)B_{14})(X,Y)\ , \end{aligned}$$

and hence

$$B^2(\mathfrak{g},\theta) = (B_{12},(\theta_1-1)B_{13} - B_{24},\theta_2 B_{24},(\theta_1-1)B_{14} - B_{23},\theta_2 B_{23}) \qquad (2.6)$$

Furthermore, B_{13} and B_{14} (which are nontrivial cocycles if $\theta_1 = 1$) are both 0 on $\mathfrak{n}_0$.

We check for which θ B_{34} is a cocycle. Thus

$$\begin{aligned} &\Sigma B_{34}(X,[Y,Z]) + \theta(X)B_{34}(Y,Z) \\ &= \Sigma(\theta_1-2)x_3(y_1 z_4-y_4 z_1) + \theta_2 x_2(y_3 z_4-y_4 z_3)\ , \end{aligned}$$

so that $B_{34} \in H^2(\mathcal{G},\theta) \Leftrightarrow \theta_1 = 2$ and $\theta_2 = 0$.

Next, we need to know $\mathrm{Aut}\,\mathcal{G}_{4,2}$.

<u>2.2.1. Lemma</u>. $\mathrm{Aut}(\mathcal{G}_{4,2})$ has four components, and is isomorphic to the group generated by the following matrices,

$$\alpha_+^+ = \begin{pmatrix} I_2 & 0_2 \\ \varphi & \begin{matrix} u & -v \\ v & u \end{matrix} \end{pmatrix}, \quad \alpha_-^+ = \begin{pmatrix} I_2 & 0_2 \\ \varphi & \begin{matrix} -u & v \\ v & u \end{matrix} \end{pmatrix}, \quad \alpha_+^- = \begin{pmatrix} \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} & 0_2 \\ \varphi & \begin{matrix} u & -v \\ v & u \end{matrix} \end{pmatrix}, \quad \alpha_-^- = \begin{pmatrix} \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} & 0_2 \\ \varphi & \begin{matrix} -u & v \\ v & u \end{matrix} \end{pmatrix}$$

where $\varphi = \begin{pmatrix} \varphi_1 & \varphi_2 \\ -\varphi_2 & \varphi_1 \end{pmatrix}$, $\varphi_1, \varphi_2 \in R$; and $u^2 + v^2 \neq 0$.

<u>Proof</u>. $\mathcal{G}_{4,2}$ is a semidirect product of $(\mathcal{G}_1)^2$ and $(\mathcal{G}_1)^2$, $\mathcal{G}_{4,2} = (\mathcal{G}_1)^2 \times_\theta (\mathcal{G}_1)^2$, where the representation θ can be realized by $\theta(e_1) = I_2$, $\theta(e_2) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\theta : Re_1 + Re_2 \to \mathrm{End}(Re_3 + Re_4)$. By [3, Corollary 2.6] any element of $\mathrm{Aut}(\mathcal{G}_{4,2})$ has a matrix representation $\alpha = \begin{pmatrix} \alpha_0 & 0_2 \\ \varphi & \psi \end{pmatrix}$, where $\varphi \in \mathrm{Hom}((\mathcal{G}_1)^2, (\mathcal{G}_1)^2)$, $\alpha_0, \psi \in GL(2,R)$, and $d\varphi = 0$, $\psi\,\theta\,\psi^{-1} = \theta \circ \alpha_0$. Letting $\alpha_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\psi = \begin{pmatrix} u & v \\ x & y \end{pmatrix}$, one obtains from the relations $\psi\,\theta(e_i)\psi^{-1} = \theta \circ \alpha_0(e_i)$, $i = 1,2$, that $a = 1$, $c = 0$, $b = 0$, $d = \pm 1$, and furthermore $x^2 = v$, $y^2 = u$.

Finally $d\varphi = \begin{pmatrix} \varphi_3 + \varphi_2 \\ \varphi_4 - \varphi_1 \end{pmatrix} B_{12}$, where $\varphi = \begin{pmatrix} \varphi_1 & \varphi_2 \\ \varphi_3 & \varphi_4 \end{pmatrix}$.

Hence $\varphi_1 = \varphi_4$ and $\varphi_3 = -\varphi_2$, and the lemma follows.

Next for $\alpha \in \mathrm{Aut}(\mathcal{G}_{4,2})$ one has

$$\alpha^t B_{34} \alpha = (-u\varphi_1 + v\varphi_2)B_{23} + (v\varphi_1 + u\varphi_2)B_{24} - (u^2+v^2)B_{34}$$

or

$$\alpha^t B_{34} \alpha = (u\varphi_1 + v\varphi_2)B_{23} + (-v\varphi_1 + u\varphi_2)B_{24} + (u^2+v^2)B_{34} ,$$

mod $B^2(\mathcal{G},\theta)$.

Hence $\mathrm{Aut}(\mathfrak{g})\times\mathrm{Aut}\,\mathfrak{a}$ acts transitively on $H^2(\mathfrak{g},\theta)$.
Observe that for $\theta = 0$, $H^2(\mathfrak{g}_{4,2}) = (B_{12})$, but $\mathfrak{d}_{B_{12}} \cap \mathfrak{z} = Re_4 \neq (0)$.
Hence we obtain no central extension of $\mathfrak{g}_{4,2}$ by $\mathfrak{g}_1$.

2.2.2. Proposition. There is within isomorphisms only one extension of $\mathfrak{g}_{4,2}$ by $\mathfrak{g}_1$, satisfying the hypothesis of Theorem 1. This is the Lie algebra $\mathfrak{g}_{5,6} = \mathfrak{g}(B_{34},\theta)$, where $\theta(e_1)e_5 = e_5$, $\ker\theta = Re_2 + Re_3 + Re_4$.

2.3. Extensions of $\mathfrak{g}_{4,3}$. The Lie algebra $\mathfrak{g} = \mathfrak{g}_{4,3}$ is defined by the brackett relations $[e_1,e_2] = e_3$, $[e_1,e_3] = e_4$, hence is nilpotent with center Re_4. Also $[\mathfrak{g},\mathfrak{g}] = Re_3 + Re_4$.

2.3.1. Let $\mathfrak{n} = (e_2,e_3,e_4)$. Then $\mathfrak{n} \approx (\mathfrak{g}_1)^3$. Put $\theta(e_1)e_5 = \tilde\theta e_5$, $\tilde\theta \in R$, and $\ker\theta = \mathfrak{n}$. We shall find the trivial cocycles w.r.t. the representation θ. Let $f \in \mathfrak{g}^*$, $f = \sum_{i=1}^{4} f_i e_i^*$. Then

$$df(X,Y) = \theta(X)f(Y) - \theta(Y)f(X) - f[X,Y]$$
$$= ((\tilde\theta f_2 - f_3)B_{12} + (\tilde\theta f_3 - f_4)B_{13} + \tilde\theta f_4 B_{14})(X,Y), \quad X,Y \in \mathfrak{g}.$$

Since $\tilde\theta \neq 0$ it follows that

$$B^2(\mathfrak{g},\theta) = (B_{12},B_{13},B_{14}) \tag{2.7}$$

Further

$$\begin{aligned}&\Sigma(B_{34}(X,[Y,Z]) + \theta(X)B_{34}(Y,Z))\\ &= \Sigma - x_4(y_1z_2 - y_2z_1) + \tilde\theta x_1(y_3z_4 - y_4z_3)\end{aligned} \tag{2.8}$$

$$\begin{aligned}&\Sigma(B_{24}(X,[Y,Z]) + \theta(X)B_{24}(Y,Z))\\ &= \Sigma(x_2(y_1z_3 - y_3z_1) + \tilde\theta x_1(y_2z_4 - y_4z_2))\end{aligned} \tag{2.9}$$

From (2.8) and (2.9) it is seen that no (nontrivial) linear combination of B_{24} and B_{34} can be a cocycle. Moreover, it is seen that

$$\begin{aligned} &\Sigma(B_{23}(X,[Y,Z]) + \theta(X)B_{23}(Y,Z)) \\ &= \Sigma\,\tilde{\theta}x_1(y_2z_3-y_3z_2) \end{aligned} \tag{2.10}$$

Thus $H^2(\mathfrak{g},\theta) = (0)$, and there is no extensions.

<u>2.3.2</u>. Let $\mathfrak{n} = (e_1,e_3,e_4)$. Hence $\mathfrak{n} \approx \mathfrak{g}_3$, the Heisenberg-algebra. Let θ be the representation of $\mathfrak{g}$ in $\mathrm{R}e_5$ defined as $\theta(e_2)e_5 = \tilde{\theta}e_5$, $\tilde{\theta} \neq 0$, and $\ker\theta = \mathfrak{n}$. Computing coboundaries we get, with $f = \sum_{i=1}^{4} f_i e_i^* \in \mathfrak{g}^*$, X, Y in $\mathfrak{g}$,

$$df(X,Y) = (-(\tilde{\theta}f_1+f_3)B_{12} + \tilde{\theta}f_3B_{23} + f_4(\tilde{\theta}B_{24}-B_{13}))(X,Y)\,.$$

Hence $B^2(\mathfrak{g},\theta) = (B_{12},B_{23},\tilde{\theta}B_{24}-B_{13})$. Next,

$$\begin{aligned} &\Sigma(B_{34}(X,[Y,Z]) + \theta(X)B_{34}(Y,Z) \\ &= \Sigma -x_4(y_1z_2-y_2z_1) + x_2(y_3z_4-y_4z_3) \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} &\Sigma(B_{14}(X,[Y,Z]) + \theta(X)B_{14}(Y,Z)) \\ &= \Sigma x_2(y_1z_4-y_4z_1) \end{aligned} \tag{2.12}$$

It follows that $H^2(\mathfrak{g},\theta) = (0)$.
Finally since any proper subalgebra of $\mathfrak{g}$, containing $[\mathfrak{g},\mathfrak{g}]$, is isomorphic to one of the subalgebras $\mathfrak{n}$ of 2.3.1 and 2.3.2, we conclude that $\mathfrak{g}$ admits no non-central extension by $\mathfrak{g}_1$.

<u>2.3.3</u>. The central extensions of $\mathfrak{g}$ by $\mathfrak{g}_1$ are all nilpotent, and are known. They are given by the cocycles B_{14} and $B_{14} + B_{23}$ (which

also form a basis for $H^2(\mathfrak{G}))$, and are denoted $\mathfrak{N}_{5,5}$ and $\mathfrak{N}_{5,6}$ respectively. Finally, if $\mathfrak{N} = [\mathfrak{G},\mathfrak{G}] = (e_3,e_4)$, then since $\dim(\mathfrak{G}/\mathfrak{N}) = 2$, any representation $\theta : \mathfrak{G} \to \mathrm{End}\, \mathfrak{G}_1$, with $\ker\theta \supset \mathfrak{N}$, has a kernel properly containing $\mathfrak{N}$, and (2.1) does not hold.

2.4. Extensions of $\mathfrak{G}_{4,4}$. The defining relations of $\mathfrak{G} = \mathfrak{G}_{4,3}$ are $[e_1,e_2] = e_3$, $[e_1,e_4] = e_4$.

2.4.1. Let $\mathfrak{N} = [\mathfrak{G},\mathfrak{G}] = (e_3,e_4)$. Hence $\mathfrak{N} \approx (\mathfrak{G}_1)^2$. Let θ be the representation of $\mathfrak{G}$ in Re_5 given as follows, $\theta(e_1)e_5 = \theta_1 e_5$, $\theta(e_2)e_5 = \theta_2 e_5$, $\ker\theta \supset \mathfrak{N}$. Let $f = \sum_{i=1}^{4} f_i e_i^* \in \mathfrak{G}^*$. We calculate $B^2(\mathfrak{G},\theta)$,

$$df = (\theta_1 f_2 - \theta_2 f_1 - f_3)B_{12} + f_3(\theta_1 B_{13} + \theta_2 B_{23}) + f_4((\theta_1 - 1)B_{14} + \theta_2 B_{24}) .$$

Hence

$$B^2(\mathfrak{G},\theta) = (B_{12}, \theta_1 B_{13} + \theta_2 B_{23}, (\theta_1 - 1)B_{14} + \theta_2 B_{24}) \qquad (2.13)$$

Also, the forms B_{13}, B_{23}, B_{14}, B_{24} are all identically zero on $\mathfrak{N}$. Checking B_{34} we have

$$B_{34}(X,[Y,Z]) + \theta(X)B_{34}(Y,Z)$$

$$= \Sigma(\theta_1 - 1)x_1(y_3z_4 - y_4z_3) + \theta_2 x_2(y_3z_4 - y_4z_3) - x_4(y_1z_2 - y_2z_1) ,$$

so that B_{34} is never a cocycle $(\theta \neq 0)$.

Therefore there is no $\mathfrak{N}$-admissible extension of $\mathfrak{G}$ by Re_5.

2.4.2. Let $\mathfrak{N} = (e_2,e_3,e_4)$. Then it follows from (2.13) with $\theta_2 = 0$ that

$$B^2(\mathfrak{G},\theta) = (B_{12}, B_{13}, (\theta_1 - 1)B_{14}) \qquad (2.14)$$

where $\theta(e_1)e_5 = \theta_1 e_5$, $\ker\theta = \mathfrak{N}$. Next, B_{34} is no cocycle by

2.4.1., and we have

$$\begin{aligned}&\Sigma B_{23}(X,[Y,Z]) + \theta(X)B_{23}(Y,Z)\\ &= \Sigma\theta_1 x_1 B_{23}(Y,Z) ,\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}&\Sigma B_{24}(X,[Y,Z]) + \theta(X)B_{24}(Y,Z)\\ &= \Sigma x_2(y_1z_4-y_4z_1) + \theta_1x_1(y_2z_4-y_4z_2)\\ &= \Sigma(1-\theta_1)x_2(y_1z_4-y_4z_1) ,\end{aligned} \tag{2.16}$$

and

$$\Sigma B_{14}(X,[Y,Z]) + \theta(X)B_{14}(Y,Z) = 0 \tag{2.17}$$

Hence $H(\mathfrak{g},\theta) = (B_{24},B_{14})$ if $\theta_1 = 1$, and is (0) otherwise. Therefore we have no $\mathfrak{n}$-admissible extensions of $\mathfrak{g}$ by $\mathfrak{g}_1$, for any $\mathfrak{n}$.

2.5. We shall treat the Lie algebras $\mathfrak{g}_{4,5}(\alpha,\beta)$, $\mathfrak{g}_{4,6}(\alpha)$, $\mathfrak{g}_{4,7}$, and $\mathfrak{g}_{4,8}(\alpha,\beta)$ together. These Lie algebras all have commutator subalgebra isomorphic to $(\mathfrak{g}_1)^3$. Let $\mathfrak{g}$ be any of the above Lie algebras, and put $\mathfrak{n} = [\mathfrak{g},\mathfrak{g}]$. Then since any alternating bilinear form B is degenerate on $\mathfrak{n}$ we have

$$\mathfrak{s}_{B^0} \cap \mathfrak{n} = \mathfrak{s}_{B^0} \cap \mathfrak{z}(\mathfrak{n}) \neq (0) ,$$

and there can be no $\mathfrak{n}$-admissible extensions of $\mathfrak{g}$ by $\mathfrak{g}_1$.

2.6. Extensions of $\mathfrak{g}_{4,9}(\alpha)$. $\mathfrak{g} = \mathfrak{g}_{4,9}(\alpha)$ is defined by the relations $[e_1,e_2] = (\alpha-1)e_2$, $[e_1,e_3] = e_3$, $[e_1,e_4] = \alpha e_4$, $[e_2,e_3] = e_4$, where $0 \leq \alpha \leq 2$, $\alpha \neq 1$, and $\varepsilon = \{e_1,e_2,e_3,e_4\}$ is a basis for the vector space of $\mathfrak{g}$.

Here nilradical and commutator subalgebra are identical,

$[\mathfrak{g},\mathfrak{g}] = \mathfrak{n}_0 = (e_2,e_3,e_4) \approx \mathfrak{g}_3$. Hence $\mathfrak{n} = \mathfrak{n}_0$ is the only possible choice for $\mathfrak{n}$ in Theorem 1. Put $\mathfrak{a} = Re_5$, and let $f : \mathfrak{g} \to Re_5$ be linear. The coboundaries are of the form

$$df(X,Y) = \theta(X)f(Y) - \theta(Y)f(X) - f[X,Y]$$
$$= \{(\tilde\theta-(\alpha-1))f_2B_{12} + (\tilde\theta-1)f_3B_{13} + ((\tilde\theta-\alpha)B_{14}-B_{23})f_4\}(X,Y) ,$$

where $\theta : \mathfrak{g} \to \mathrm{End}\,\mathfrak{a}$, $\ker\theta = \mathfrak{n}$, $\theta(e_1)e_5 = \tilde\theta e_5$. Hence the trivial cocycles are

$$((\tilde\theta-(\alpha-1))B_{12},(\tilde\theta-1)B_{13},(\tilde\theta-\alpha)B_{14}-B_{23}) = B^2(\mathfrak{g},\theta) .$$

We remark that B_{12} is nontrivial for $\tilde\theta = \alpha-1$, and B_{13} for $\tilde\theta = 1$.

Next $\mathfrak{z}_{B^0} \cap \mathfrak{z} = (0)$ only for forms B in (B_{24},B_{34}). We shall determine for which θ-s these forms are cocycles.

Consider first B_{34} :

$$\Sigma(B_{34}(X,[Y,Z]) - \theta(X)B_{34}(Y,Z))$$
$$= \Sigma(\alpha+1-\tilde\theta)x_3(y_1z_4-y_4z_1) .$$

<u>2.6.1. Lemma</u>. Aut $\mathfrak{g}_{4,9}(\alpha)$ is isomorphic to the group of all real matrices A, where

(a) $\alpha \neq 0,2$:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \varphi_1 & a & 0 & 0 \\ \varphi_2 & 0 & b & 0 \\ \varphi_3 & a\varphi_2 & \psi & ab \end{pmatrix} , \text{ where } ab \neq 0 \text{ and } \psi = \varphi_1\, b/_{1-\alpha}$$

(b) $\alpha = 0$:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \varphi_1 & a & 0 & 0 \\ \varphi_2 & 0 & b & 0 \\ \varphi_3 & a\varphi_2 & b\varphi_1 & ab \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -1 & 0 & 0 & 0 \\ \varphi_1 & 0 & a & 0 \\ \varphi_2 & b & 0 & 0 \\ \varphi_3 & -b\varphi_1 & -a\varphi_2 & -ab \end{pmatrix} \quad , \text{ where } \quad ab \neq 0$$

(c) $\alpha = 2$:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \varphi_1 & a & b & 0 \\ \varphi_2 & c & d & 0 \\ \varphi_3 & a\varphi_2 - c\varphi_1 & b\varphi_2 - d\varphi_1 & c_0 \end{pmatrix} \quad , \text{ where } \quad c_0 = ad-bc \neq 0 \; .$$

<u>Proof</u>. If $A \in \operatorname{Aut} \mathcal{G}$ then $A|\mathcal{N}_0 \in \operatorname{Aut}(\mathcal{N}_0)$ where $\mathcal{N}_0$ denotes the nilradical of $\mathcal{G}$, $\mathcal{N}_0 = (e_2, e_3, e_4) \approx \mathcal{G}_3$. Hence $A|\mathcal{N}_0$ is of the form $\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ u & v & c_0 \end{pmatrix}$, $c_0 = ad-bc \neq 0$. Therefore we may assume that A has a matrixrepresentation

$$\begin{pmatrix} \alpha_0 & 0 & 0 & 0 \\ \varphi_1 & a & b & 0 \\ \varphi_2 & c & d & 0 \\ \varphi_3 & u & v & c_0 \end{pmatrix} \quad , \quad \alpha_0 \neq 0 \; .$$

Hence B_{34} satisfies the cocycle identity iff $\tilde{\theta} = 1+\alpha$.

Next consider B_{24} :

$$\Sigma B_{24}(X,[Y,X]) - \theta(x) B_{24}(Y,Z)$$

$$= \Sigma (2\alpha - 1 - \tilde{\theta}) x_1 (y_2 z_4 - y_4 z_2)$$

So that B_{24} is a cocycle iff $\tilde{\theta} = 2\alpha - 1$.

From this it follows that $(B_{24}, B_{34}) \subset H^2(\mathcal{G}, \theta)$ iff $\tilde{\theta} = 3$.

Furthermore it is seen that $B_{12} \in H^2(\mathcal{G}, -1)$ and $B_{13} \in H^2(\mathcal{G}, 1)$, and for $\tilde{\theta} \neq \pm 1$ B_{12} and B_{13} fail to be cocycles. We conclude that

$$H^2(\mathfrak{g}, 2\alpha-1) = \begin{cases} (B_{24}), & \alpha \neq 0,2 \\ (B_{24}, B_{12}), & \alpha = 0 \\ (B_{24}, B_{34}), & \alpha = 2 \end{cases} \tag{2.18}$$

$$H^2(\mathfrak{g}, \alpha+1) = \begin{cases} (B_{34}), & \alpha \neq 0,2 \\ (B_{34}, B_{13}), & \alpha = 0 \\ (B_{24}, B_{34}), & \alpha = 2 \end{cases} \tag{2.19}$$

Calculations give

$$[AX, AY] = \begin{bmatrix} 0 \\ (\alpha-1)\alpha_0(aB_{12}+bB_{13}) \\ \alpha_0(cB_{12}+dB_{13}) \\ (\alpha\alpha_0 u+\varphi_1 c-\varphi_2 a)B_{12} + (\alpha\alpha_0 v+\varphi_1 d-\varphi_2 b)B_{13}+\alpha\alpha_0 c_0 B_{14}+c_0 B_{23} \end{bmatrix} (X,Y)$$

and

$$A[X,Y] = \begin{bmatrix} 0 \\ (\alpha-1)aB_{12} + bB_{13} \\ (\alpha-1)cB_{12} + dB_{13} \\ (\alpha-1)uB_{12} + vB_{13} + c_0 \; B_{14} + c_0 B_{23} \end{bmatrix} (X,Y)$$

Letting $A[X,Y] = [AX,AY]$ the lemma follows.

<u>Remark</u>. Aut $\mathfrak{g}_{4,9}(\alpha)$ is solvable unless $\alpha = 2$.

Next we compute Aut $\mathfrak{g}$ - orbits in $\bigcup_\theta H^2(\mathfrak{g}, \theta)$.

<u>2.6.2. Lemma</u>

(a) Aut $\mathfrak{g}_{4,9}(2)$ acts transitively on

$$H^2(\mathfrak{g}_{4,9}(2), 3) = (B_{24}, B_{34})$$

(b) For $\alpha = 0$, the orbit of B_{24} under Aut $\mathfrak{g}$ is

$$H^2(\mathfrak{g}, -1) \cup H^2(\mathfrak{g}, 1) = (B_{12}, B_{24}) \cup (B_{13}, B_{34})$$

(c) For $\alpha \neq 0,2$, $\tilde{\theta} = 2\alpha-1$, we have

$$(\mathrm{Aut}\,\mathcal{G})\circ B_{24} = H^2(\mathcal{G}, 2\alpha-1)$$

(d) For $\alpha \neq 0,2$, $\tilde{\theta} = \alpha+1$, we have

$$(\mathrm{Aut}\,\mathcal{G})\circ B_{34} = H^2(\mathcal{G}, \alpha+1)$$

Proof. (a) Let $\alpha = 2$, $\tilde{\theta} = 3$. Then

$$A^t B_{24} A = ac_0 B_{24} + bc_0 B_{34}\,, \quad \mathrm{mod}\ B^2(\mathcal{G},3)\,,$$

where A is as in Lemma 2.6.2. (c).

(b) Let $\alpha = 0$, $\tilde{\theta} = -1$. Then

$$A^t B_{24} A = -a(\varphi_3 - \varphi_1\varphi_2)B_{12} + a^2 b B_{24}\,, \quad \mathrm{mod}\ B^2(\mathcal{G},-1)\,,$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \varphi_1 & a & 0 & 0 \\ \varphi_2 & 0 & b & 0 \\ \varphi_3 & a\varphi_2 & b\varphi_1 & ab \end{pmatrix}\,, \quad ab \neq 0\,,$$

and

$$A_0^t B_{24} A_0 = a(\varphi_3 + \varphi_1\varphi_2)B_{13} - a^2 b B_{34}\,, \quad \mathrm{mod}\ B^2(\mathcal{G},1)\,,$$

where

$$A_0 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} A\,, \quad \text{and} \quad \theta \circ A_0 = -\theta\,.$$

Hence $\mathrm{Aut}(\mathcal{G}_{4,9}(0))\circ B_{24} = (B_{12}, B_{24}) \cup (B_{13}, B_{34})$.

(c) $\alpha \neq 0,2$, $\tilde{\theta} = 2\alpha-1$. Note first that $\theta \circ A = \theta$, for all $A \in \mathrm{Aut}\,\mathcal{G}$, see Lemma 2.6.2. (a). Hence

$$(\mathrm{Aut}\,\mathcal{G})B_{24} = H^2(\mathcal{G}, 2\alpha-1)$$

(d) follows as (c).

Combining the results in Lemma 2.6.2 we obtain

2.6.3. Proposition. Let $\mathfrak{g} = \mathfrak{g}_{4,9}(\alpha)$, $0 \leq \alpha \leq 2$, $\alpha \neq 1$, and put $\mathfrak{n} = [\mathfrak{g},\mathfrak{g}]$. The admissible extensions of $\mathfrak{g}$ by $\mathfrak{g}_1$ are within isomorphisms, the following,

$$\mathfrak{g}_{5,7}(\alpha) = \mathfrak{g}(B_{24}, 2\alpha-1), \text{ where } 0 < \alpha \leq 2, \alpha \neq 1.$$

and

$$\mathfrak{g}_{5,8}(\alpha) = \mathfrak{g}(B_{34}, \alpha+1), \text{ where } 0 \leq \alpha \leq 2, \alpha \neq 1.$$

These Lie algebras are pairwise non-isomorphic.

2.7. Extensions of $\mathfrak{g}_{4,10}$. Let $\mathfrak{g} = \mathfrak{g}_{4,10}$. The defining relations of $\mathfrak{g}$ are $[e_1,e_2] = e_2+e_3$, $[e_1,e_3] = e_3$, $[e_1,e_4] = 2e_4$. Hence $[\mathfrak{g},\mathfrak{g}] = (e_2,e_3,e_4)$ and is equal to the nilradical. Thus we let $\mathfrak{n} = (e_2,e_3,e_4)$. Let $f = \sum_{i=1}^{4} f_i e_i^* \in \mathfrak{g}^*$. Then

$$df(X,Y) = \theta(X)f(Y) - \theta(Y)f(X) - f[x,y]$$
$$= (((\tilde{\theta}-1)f_2 - f_3)B_{12} + (\tilde{\theta}-1)f_3B_{13} + f_4((\tilde{\theta}-2)B_{14} - B_{23}))(X,Y),$$

where $\theta(e_1)e_5 = \tilde{\theta}e_5$, $\tilde{\theta} \neq 0$, $\ker\theta = \mathfrak{n}$, defines a representation θ of $\mathfrak{g}$ in $Re_5 = \mathfrak{g}_1$. Hence the space of coboundaries is

$$B^2(\mathfrak{g},\theta) = (B_{12}, (\tilde{\theta}-1)B_{13}, (\tilde{\theta}-2)B_{14} - B_{23}) \tag{2.20}$$

Furthermore,

$$\begin{aligned}&\Sigma B_{24}(X,[Y,Z]) + \theta(X)B_{24}(Y,Z)\\ &= (3-\tilde{\theta})\, \Sigma x_2(y_1z_4 - y_4z_1)\end{aligned} \tag{2.21}$$

and

$$\begin{aligned}&\Sigma B_{34}(X,[Y,Z]) + \theta(X)B_{34}(Y,Z)\\ &= \Sigma(3-\tilde{\theta})x_3(y_1z_4 - y_4z_1) - x_4(y_1z_2 - y_2z_1)\end{aligned} \tag{2.22}$$

In addition B_{13} and B_{14} are seen to satisfy the cocycle identity for all $\tilde{\theta}$. It follows from this together with (2.20), (2.21), and (2.22) that

$$H^2(\mathfrak{g},\theta) = \begin{cases} (B_{13}) \;, & \tilde{\theta} = 1 \\ (B_{24}) \;, & \tilde{\theta} = 3 \\ (B_{14}) \;, & \tilde{\theta} = 2 \\ (0) \;, & \tilde{\theta} \notin \{1,2,3\} \end{cases} \tag{2.23}$$

Now B_{13} and B_{14} vanish on $\mathfrak{n}$, whereas $\mathfrak{s}_{B_{24}} \cap \mathrm{Re}_4 = (0)$. Hence we obtain $\mathfrak{n}$-admissible extensions of $\mathfrak{g}$ by $\mathfrak{g}_1$ only when $\tilde{\theta} = 3$. Obviously $\mathrm{Aut}\,\mathfrak{g} \times \mathrm{Aut}(\mathrm{Re}_5)$ acts transitively on (B_{24}) ($\mathrm{Aut}(\mathrm{Re}_5)$ acts by multiplication with a nonzero scalar), and we have only one isomorphism-class of extensions.

<u>2.7.1. Proposition</u>. The only admissible extension of $\mathfrak{g} = \mathfrak{g}_{4,10}$ by $\mathfrak{g}_1$ is the solvable Lie algebra $\mathfrak{g}_{5,9} = \mathfrak{g}(B_{24},3)$.

<u>2.8. Extensions of $\mathfrak{g}_{4,11}(\alpha)$</u>. Let $\mathfrak{g} = \mathfrak{g}_{4,11}(\alpha)$, $\alpha \geq 0$. $\mathfrak{g}$ is defined by the relations $[e_1,e_2] = \alpha e_2 - e_3$, $[e_1,e_3] = e_2 + \alpha e_3$, $[e_1,e_4] = 2\alpha e_4$, $[e_2,e_3] = e_4$. Again $[\mathfrak{g},\mathfrak{g}] = (e_2,e_3,e_4) \approx \mathfrak{g}_3$, and is equal to the nilradical. Hence we let $\mathfrak{n} = (e_2,e_3,e_4)$. Let $f = \sum_{i=1}^{4} f_i e_i^* \in \mathfrak{g}^*$, and define a representation of $\mathfrak{g}$ in $\mathfrak{g}_1 = \mathrm{Re}_5$ by $\theta(e_1)e_5 = \tilde{\theta}e_5$, $\ker\theta = \mathfrak{n}$. Then

$$\begin{aligned} df(X,Y) &= \theta(X)f(Y) - \theta(Y)f(X) - f[X,Y] \\ &= ((\tilde{\theta}-\alpha)B_{12}-B_{13})f_2 + f_3((\tilde{\theta}-\alpha)B_{13}+B_{12} + f_4((\tilde{\theta}-2\alpha)B_{14}-B_{23})(X,Y) \end{aligned} \tag{2.24}$$

and the space of coboundaries is

$$B^2(\mathfrak{g},\theta) = ((\tilde{\theta}-\alpha)B_{13} + B_{12}, (\tilde{\theta}-\alpha)B_{12} - B_{13}, (\tilde{\theta}-2\alpha)B_{14} - B_{23})$$

Also

$$\Sigma B_{24}(X,[Y,Z]) + \theta(X)B_{24}(Y,Z) \\ = \Sigma(\tilde{\theta}-3\alpha)x_4(y_1z_2-y_2z_1) - x_4(y_1z_2-y_2z_1) , \tag{2.25}$$

and

$$\Sigma B_{34}(X,[Y,Z]) + \theta(x)B_{34}(Y,Z) \\ = \Sigma(3\alpha-\tilde{\theta})x_3(y_1z_4-y_4z_1) + x_4(y_1z_2-y_2z_1) \tag{2.26}$$

From (2.25) and (2.26) we see that $sB_{24} + tB_{34}$ is a cocycle iff $(\tilde{\theta}-3\alpha)s+t = 0$ and $s + (3\alpha-\tilde{\theta})t = 0$, and this system has only the solution $s = t = 0$, for all $\tilde{\theta}$. It follows that $H^2(\mathfrak{g},\theta) = (0)$ for all θ and all α.

Hence there is no (admissible) extension of $\mathfrak{g}_{4,10}$ by $\mathfrak{g}_1$.

2.8. In this section we shall study extensions of reducible solvable Lie algebras $\mathfrak{g}$ of dimension four, that is, Lie algebras which are direct products of proper subalgebras.

2.8.1. Let $\mathfrak{g}$ be the abelian Lie algebra $(\mathfrak{g}_1)^4$ with basis $(e_i)_{i=1}^4$, and let $\theta : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}_1)$ be a representation with $\ker\theta \supset \mathfrak{n}$, $\mathfrak{n}$ being a subalgebra of $\mathfrak{g}$.

Now if $\mathfrak{n} = \mathfrak{g}$, then $\theta = 0$ and any corresponding extension of $\mathfrak{g}$ by $\mathfrak{g}_1$ will be nilpotent. As is well known the only (admissible) nilpotent extension of $\mathfrak{g}$ by $\mathfrak{g}_1$ is given by the cocycle $B = B_{12} + B_{34}$. This Lie algebra is denoted $\mathfrak{n}_{5,1}$.

Next if $\mathfrak{g} \approx (\mathfrak{g}_1)^3$, assume $\mathfrak{n} = (e_1,e_2,e_3)$. For any alternating bilinear form B on $\mathfrak{g}$ we have $\mathfrak{z}_{B^\circ} \cap \mathfrak{z} = \mathfrak{z}_{B^\circ} \cap \mathfrak{n} \neq (0)$. Hence there is no $\mathfrak{n}$-admissible extension.

Finally, if $\mathfrak{n}$ is isomorphic with one of the following Lie algebras,

(0), $\mathcal{G}_1$, $(\mathcal{G}_1)^2$, then there is $X \in \mathcal{G} - \mathcal{N}$, $X \neq 0$, with $\theta(X) = 0$. Thus $\theta(X)$ is nilpotent, and we have no $\mathcal{N}$-admissible extensions.

2.8.2. Let $\mathcal{G} = \mathcal{G}_2 \times (\mathcal{G}_1)^2$ with basis $(e_i)_{i=1}^4$, and defining Lie products $[e_1,e_2] = e_2$. In this case the nilradical is $\mathcal{N}_0 = (e_2,e_3,e_4) \approx (\mathcal{G}_1)^3$ and $[\mathcal{G},\mathcal{G}] = (e_2)$. Now only $\mathcal{N} = (e_2,e_3)$ (or (e_2,e), $e \in \mathcal{N}_0$, $e \neq e_2$) can give any $\mathcal{N}$-admissible extensions. For if $\mathcal{N} = (e_2)$ or $\mathcal{N} = (e_2,e_3,e_4)$ then $\mathcal{S}_{B^0} \cap \mathcal{Z}(\mathcal{N}) \neq (0)$ where B is an arbitrary alternating bilinear form on $\mathcal{G}$. Hence let $\mathcal{N} = (e_2,e_3)$, and define $\theta : \mathcal{G} \to \mathrm{End}(\mathcal{G}_1)$ by $\theta(e_1)e_5 = \theta_1 e_5$, $\theta(e_4)e_5 = \theta_4 e_4$, and $\ker\theta \supset \mathcal{N}$. Here we assume $\theta_4 \neq 0$ since $e_4 \in \mathcal{N}_0$, cfr. (1.2). Let $f = \sum_{i=1}^{4} f_i e_i^* \in \mathcal{G}^*$. Then

$$
\begin{aligned}
&\Sigma B_{23}(X,[Y,Z]) + \theta(X)B_{23}(Y,Z) \\
&= \Sigma(\theta_1 - 1)x_1(y_2z_3 - y_3z_2) + \theta_4x_4(y_2z_3 - y_3z_2)
\end{aligned}
\tag{2.27}
$$

and

$$
df = ((\theta_1-1)B_{12} - \theta_4B_{24})f_2 + (f_4 - \theta_4f_1)B_{14} + f_3(\theta_1B_{13} - \theta_4B_{34})
$$

Hence

$$
B^2(\mathcal{G},\theta) = ((\theta_1-1)B_{12} - \theta_4B_{24}, B_{14}, \theta_1B_{13} - \theta_4B_{34}) \tag{2.28}
$$

It follows easily from (2.27) and (2.28) that $H^2(\mathcal{G},\theta) = (0)$, all θ. Hence we have no extensions of $\mathcal{G}_2 \times (\mathcal{G}_1)^2$ by $\mathcal{G}_1$.

2.8.3. Let $\mathcal{G} = \mathcal{G}_2 \times \mathcal{G}_2$ with basis $(e_i)_{i=1}^4$. The defining relations for $\mathcal{G}$ are $[e_1,e_2] = e_2$, $[e_3,e_4] = e_4$. Here $\mathcal{N}_0 = [\mathcal{G},\mathcal{G}] = (e_2,e_4) \approx (\mathcal{G}_1)^2$. Let $\theta : \mathcal{G} \to \mathrm{End}(\mathcal{G}_1)$ be given by $\theta(e_1)e_5 = \theta_1e_5$, $\theta(e_3)e_5 = \theta_3e_5$ and $\ker\theta \supset \mathcal{N}_0$. Let $B = B_{24}$.

Then $\mathfrak{s}_{B^o} \cap \mathfrak{z} = (0)$, and

$$\Sigma B_{24}(X,[Y,Z]) + \theta(X)B_{24}(Y,Z)$$
$$= \Sigma(1-\theta_1)x_1(y_2z_4-y_4z_2) + (1-\theta_3)x_2(y_3z_4-y_4z_3)$$

so that B_{24} is a cocycle iff $\theta_1 = \theta_3 = 1$.
Computing coboundaries in this case we find, for $f = \sum_{i=1}^{4} f_i e_i^* \in \mathfrak{g}^*$,

$$df = (f_3-f_1)B_{13} + f_4B_{14} - f_2B_{23} ,$$

hence $B^2(\mathfrak{g},(1,1)) = (B_{13},B_{14},B_{23})$. It is seen that $aB_{12} + bB_{34}$ is never a cocycle $((a,b) \neq (0,0))$.
Therefore

$$H^2(\mathfrak{g},(1,1)) = (B_{24}) ,$$

and we have only one admissible extension, namely $\mathfrak{g}(B_{24},(1,1)) = \mathfrak{g}_{5,1}$.

<u>2.8.4</u>. Let $\mathfrak{g} = \mathfrak{g}_{3,1} \times \mathfrak{g}_1$ with defining relations $[e_1,e_2] = e_3$, $\mathfrak{g}_1 = Re_4$. $\mathfrak{g}$ is nilpotent and the commutator is $[\mathfrak{g},\mathfrak{g}] = (e_3)$. Now the admissible central extension of $\mathfrak{g}$ by $\mathfrak{g}_1$ (corresponding to $\mathfrak{n} = \mathfrak{g}$ and $\theta = 0$) are known to be isomorphic with the nilpotent Lie algebra $\mathfrak{n}_{5,3}$ given by the cocycle $B = B_{14} + B_{23}$.

<u>2.8.4.1</u>. Assume $\mathfrak{n} \subset \mathfrak{g}$ and $\mathfrak{n} \approx \mathfrak{g}_3$, the 3-dimensional Heisenberg algebra. Via a change of basis we may assume $\mathfrak{n} = (e_1,e_2,e_3)$. Now let $\theta : \mathfrak{g} \to \mathrm{End}(Re_5)$ be a representation, and let $\theta(e_1)e_5 = \tilde{\theta}e_5$, $\ker\theta = \mathfrak{n}$. If $f = \sum_{i=1}^{4} f_i e_i^* \in \mathfrak{g}^*$ we have

$$df(X,Y) = -(f_3(B_{12}+\theta B_{34})+\theta f_1B_{14}+f_2B_{24})(X,Y)$$

Hence the coboundaries are

$$B^2(\mathfrak{g},\theta) = (B_{12}+\theta B_{34},B_{14},B_{24}) \qquad (2.29)$$

Further, B_{13} and B_{23} are seen not to span any cocycle. Hence $H^2(\mathfrak{g},\theta) = (0)$ for all θ ; $\theta \neq 0$, and there is no $\mathfrak{n}$-admissible extension of $\mathfrak{g}$.

2.8.4.2. Suppose $\mathfrak{n} \subset \mathfrak{g}$, $\mathfrak{n} \approx (\mathfrak{g}_1)^3$. In this case $\mathfrak{s}_{B^\circ} \cap \mathfrak{z}(\mathfrak{n}) = \mathfrak{s}_{B^\circ} \cap \mathfrak{n} \neq (0)$ for every cocycle B, so there is no $\mathfrak{n}$-admissible extension. Finally, if $\dim \mathfrak{n} = 2$ or $\dim \mathfrak{n} = 1$, then for any representation $\theta : \mathfrak{g} \to \mathrm{End}(Re_5)$ with $\ker\theta \supset \mathfrak{n}$, we have $\ker\theta \neq \mathfrak{n}$, and since $\mathfrak{g}$ is nilpotent there is no $\mathfrak{n}$-admissible extension, (1.2).

2.8.5. Arguments similar to those of 2.8.4 show that $\mathfrak{g}_{3,3} \times \mathfrak{g}_1$ and $\mathfrak{g}_{3,4}(\alpha) \times \mathfrak{g}_1$ permit no $\mathfrak{n}$-admissible extension by $\mathfrak{g}_1$, for any nilpotent subalgebra $\mathfrak{n}$ containing $[\mathfrak{g},\mathfrak{g}]$.

2.8.6. We summarize the above in the following

Proposition. Let $\mathfrak{g}$ be a reducible solvable Lie algebra of dimension four over R, and let $\mathfrak{n}$ be a nilpotent subalgebra of $\mathfrak{g}$ containing $[\mathfrak{g},\mathfrak{g}]$. If $\tilde{\mathfrak{g}}$ is a non-nilpotent $\mathfrak{n}$-admissible extension of $\mathfrak{g}$ by $\mathfrak{g}_1 = Re_5$ then

(1) $\mathfrak{g} \approx \mathfrak{g}_2 \times \mathfrak{g}_2$ with basis relations $[e_1,e_2] = e_2$, $[e_3,e_4] = e_4$,

and

(2) $\tilde{\mathfrak{g}} \approx \mathfrak{g}(B_{24},\theta)$ where $\theta(e_1)e_5 = e_5$, $\theta(e_3)e_5 = e_5$, and $\ker\theta = (e_2,e_4) \approx \mathfrak{n}$. This Lie algebra is denoted $\mathfrak{g}_{5,1}$.

3. Extensions of three dimensional solvable Lie algebras

The present chapter is devoted to the extensions of three dimensional solvable Lie algebras $\mathfrak{g}$ by $\mathfrak{a} = (\mathfrak{g}_1)^2$. We start with the irreducible cases.

3.1. Extensions of $\mathfrak{g}_{3,1}$. $\mathfrak{g} = \mathfrak{g}_{3,1}$ is nilpotent and is given by the relations $[e_1,e_2] = e_3$, $(e_i)_{i=1}^{3}$ being a basis for $\mathfrak{g}$. This is the three-dimensional Heisenberg-algebra. Here $[\mathfrak{g},\mathfrak{g}] = (e_3)$.

3.1.1. Let $\dim \mathfrak{n} = 2$, $\mathfrak{n} \subset \mathfrak{g}$. We may assume $\mathfrak{n} = (e_2,e_3)$. Let $\theta : \mathfrak{g} \to \mathrm{End}(Re_4+Re_5)$ be a representation, where we assume $\mathfrak{a} = Re_4+Re_5$. Suppose $\ker\theta = \mathfrak{n}$. Thus θ is uniquely given by $\theta(e_1)$, which we realize as a 2×2-matrix relative to the basis (e_4,e_5). We assume $\theta(e_1)$ is of Jordan normal form since $\Psi\theta\Psi^{-1}$ and θ give isomorphic Lie algebras, where $\Psi \in \mathrm{Aut}(\mathfrak{a}) = GL(2,R)$. Next, we shall need the following known fact,

3.1.1.1. Lemma. The automorphism group $\mathrm{Aut}(\mathfrak{g}_{3,1})$ is canonically isomorphic with the group of all matrices

$$A = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ \varphi_1 & \varphi_2 & e \end{pmatrix}, \text{ where } ad-bc = e \neq 0 .$$

3.1.1.2. Hence if $A \in \mathrm{Aut}(\mathfrak{g}_{3,1})$ as above, we have $\theta(Ae_1) = (a+c+\varphi_1)\theta(e_1)$, and if $\theta(e_1) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, $\beta \neq 0$, then $\theta(Ae_1) = \begin{pmatrix} \alpha/\beta & 0 \\ 0 & 1 \end{pmatrix}$ where we put $a = \beta^{-1}$, $c = b = \varphi_1 = \varphi_2 = 0$.
Thus we may assume $\theta(e_1) = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$, $\alpha \in R$. Calculating coboundaries one finds

$$B^2(\mathfrak{g}_{1,3},\theta) = (B_{12}e_4, B_{13}e_4, B_{12}e_5, B_{13}e_5) \tag{3.1}$$

Moreover, $B_{23}e_5$ doesn't satisfy the cocycle identity for any α, whereas $B_{23}e_4 \in H^2(\mathcal{G},\theta)$ if $\alpha = 0$. Therefore we have one $\mathcal{N}$-admissible extension,

$$\mathcal{G}(\begin{pmatrix} B_{23} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) = \mathcal{G}_{5,15} .$$

<u>3.1.1.3</u>. Next, assume $\theta(e_1)$ has complex eigenvalues.

Applying a suitable Ψ in $GL(2,R)$ we may assume $\theta(e_1) = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, where $\beta \neq 0$ and considering (Ae_1) where $A = \begin{pmatrix} \beta^{-1} & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Aut($\mathcal{G}$), we assume $\theta(e_1) = \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix}$. Letting $f = \varphi e_4 + \psi e_5$, where $\varphi, \psi \in \mathcal{G}^*$, $\varphi = \sum_{i=1}^{3} \varphi_i e_i^*$, $\psi = \sum_{i=1}^{3} \psi_i e_i^*$, we find

$$df = \begin{pmatrix} (\alpha\varphi_2 + \psi_2 - \varphi_3)B_{12} + (\alpha\varphi_3 + \psi_3)B_{13} \\ (\alpha\psi_2 - \psi_3 - \varphi_2)B_{12} + (\alpha\psi_3 - \varphi_3)B_{13} \end{pmatrix}$$

and therefore

$$B^2(\mathcal{G},\theta) = (B_{12}e_4, B_{12}e_5, B_{13}e_4, B_{13}e_5) \tag{3.2}$$

Moreover,

$$\begin{aligned} &\Sigma \begin{pmatrix} s \\ t \end{pmatrix} B_{23}(X[Y,Z]) + \theta(X) \begin{pmatrix} s \\ t \end{pmatrix} B_{23}(Y,Z) \\ &= \Sigma x_1 \begin{pmatrix} \alpha s + t \\ \alpha t - s \end{pmatrix} B_{23}(Y,Z) , \end{aligned} \tag{3.3}$$

and hence $\begin{pmatrix} s \\ t \end{pmatrix} B_{23} \notin H^2(\mathcal{G},\theta)$ for any $\begin{pmatrix} s \\ t \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

It follows that $H^2(\mathcal{G},\theta) = (0)$.

<u>3.1.1.4</u>. Assume $\theta(e_1)$ has exactly one (real) eigenvalue.

Conjugating with a suitable ψ in $GL(2,R)$ we may let $\theta(e_1) = \begin{pmatrix} a & a \\ 0 & a \end{pmatrix}$, and applying $A = \begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix}$ in Aut($\mathcal{G}$) we finally put $\theta(e_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, or $\theta(e_1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in case $a = 0$.

The latter matrix is nilpotent, and there is no corresponding $\mathcal{N}$-admissible extension by (1.2). Put $\theta(e_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. It is seen that

$$B^2(\mathfrak{g},\theta) = (B_{12}e_4, B_{13}e_4, B_{12}e_5, B_{13}e_5) \tag{3.4}$$

Moreover, $\binom{s}{t}B_{23} \notin H^2(\mathfrak{g},\theta)$, all s,t, hence $H^2(\mathfrak{g},\theta) = (0)$, and we have no $\mathfrak{n}$-admissible extension.

3.1.2. Let $\mathfrak{n} = \mathfrak{g}_{3,1}$, hence $\theta = 0$. The irreducible central extensions of $\mathfrak{g}_{3,1}$ by $(\mathfrak{g}_1)^2$ are known, and are all isomorphic with $\mathfrak{n}_{5,4} = \left(\begin{bmatrix} B_{13} \\ B_{14} \end{bmatrix}\right)$.

Finally, if $\mathfrak{n} = (e_3)$ we have $\mathfrak{s}_{B^0} \cap \mathfrak{z}(\mathfrak{n}) \neq (0)$ for all alternating forms B on $\mathfrak{g}$, and there is no $\mathfrak{n}$-admissible extension.

3.2. Extensions of $\mathfrak{g}_{3,2}(\alpha)$. $\mathfrak{g} = \mathfrak{g}_{3,2}(\alpha)$, $|\alpha| \geq 1$, is given by the nonzero basis relations $[e_1,e_2] = e_2$, $[e_1,e_3] = \alpha e_3$. Hence the nilradical is equal to the commutator subalgebra, $\mathfrak{n}_0 = [\mathfrak{g},\mathfrak{g}] = (e_2,e_3)$. Hence $\mathfrak{n} = [\mathfrak{g},\mathfrak{g}]$. From [3, Cor.2.6] one easily derives the following

3.2.1. Lemma. Aut $\mathfrak{g}_{3,2}(\alpha)$ is canonically isomorphic to the group of all matrices A where

(1) $$A = \begin{bmatrix} 1 & 0 & 0 \\ \varphi_1 & a & 0 \\ \varphi_2 & 0 & d \end{bmatrix}, \quad ad \neq 0, \quad \text{whenever} \quad \alpha \neq 1.$$

(2) $$A = \begin{bmatrix} 1 & 0 & 0 \\ \varphi_1 & a & b \\ \varphi_2 & c & d \end{bmatrix}, \quad ad-bc \neq 0, \quad \text{whenever} \quad \alpha = 1.$$

<u>3.2.2</u>. Assume $\theta(e_1) = \begin{pmatrix} \beta & 0 \\ 0 & \gamma \end{pmatrix}$. Let $f = \begin{pmatrix} \varphi \\ \psi \end{pmatrix} : \mathcal{G} \to \mathfrak{A} = Re_4 + Re_5$ be linear, where $\varphi = \sum_{i=1}^{3} \varphi_i e_i^*$, $\psi = \sum_{i=1}^{3} \psi_i e_i^* \in \mathcal{G}^*$, we see that

$$df = \begin{pmatrix} \varphi_2(\gamma-1)B_{12} + (\gamma\varphi_2-\alpha\varphi_3)B_{13} \\ \psi_2(\beta-1)B_{12} + (\beta\psi_2-\alpha\psi_3)B_{13} \end{pmatrix} \tag{3.5}$$

Further

$$\Sigma\binom{r}{s}B_{23}(X,[Y,Z]) + \theta(X)\binom{r}{s}B_{23}(Y,Z)$$
$$= \Sigma x_1\binom{r(\alpha+1)-r\beta}{s(\alpha+1)-s\gamma}B_{23}(Y,Z) \tag{3.6}$$

Combining (3.5) and (3.6) we find, omitting symmetric cases, that $H^2(\mathcal{G}_{3,2}(\alpha),\theta)$ is generated by the following cocycles,

$$\begin{pmatrix} B_{23} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ B_{23} \end{pmatrix}, \quad \beta = \gamma = \alpha+1 \tag{3.7}$$

$$\begin{pmatrix} B_{23} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} B_{12} \\ 0 \end{pmatrix}, \quad \beta = \alpha+1, \quad \gamma = 1 \tag{3.8}$$

$$\begin{pmatrix} B_{23} \\ 0 \end{pmatrix}, \quad \beta = \alpha+1, \quad \gamma \neq 1 \tag{3.9}$$

and $H^2(\mathcal{G}_{3,2}(\alpha),\theta)$ concists of cocycles B with $\mathcal{S}_{B^0} \cap \mathfrak{Z}(\mathcal{N}) \neq (0)$ otherwise.

It follows from Lemma 3.2.1. that $\theta \circ A = \theta$ for every A in $\mathrm{Aut}(\mathcal{G})$. Hence we can not simplify $\theta(e_1)$ as in 3.1.1.2. Next we compute the action of $\mathrm{Aut}(\mathcal{G})$ in $H^2(\mathcal{G},\ )$. Thus, for $A \in \mathrm{Aut}(\mathcal{G})$,

$$A^tB_{12}A = \begin{cases} aB_{12} + bB_{13}, & \alpha = 1 \\ aB_{12}, & \alpha \neq 1 \end{cases} \tag{3.10}$$

and

$$A^tB_{23}A = \begin{cases} (ad-bc)B_{23} + (c\varphi-a\varphi_2)B_{12} + (d\varphi_1-b\varphi_2)B_{13}, & \alpha = 1 \\ ad\,B_{23} - a\varphi_2\,B_{12} + d\varphi_1\,B_{13}, & \alpha \neq 1 \end{cases} \tag{3.11}$$

It follows from (3.10) and (3.11) that $\mathrm{Aut}(\mathcal{G})$ acts transitively on $H^2(\mathcal{G},\theta)$ in all the three cases (3.7) - (3.9), and since

$\begin{pmatrix} B_{23} \\ 0 \end{pmatrix} \in H^2(\mathcal{G},\theta)$ in all these cases, we obtain the following family of extensions:

$$\mathcal{G}_{5,17}(\alpha,\beta) = \mathcal{G}\left(\begin{pmatrix} B_{23} \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha+1 & 0 \\ 0 & \beta \end{pmatrix}\right), \qquad |\alpha| \geq 1,\ \beta \neq 0 .$$

<u>3.2.3</u>. Let $\theta(e_1) = \begin{pmatrix} \beta & 1 \\ 0 & \beta \end{pmatrix}$, and let $f = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ as in 3.2.2. Then we find

$$df = \begin{pmatrix} (\beta-\alpha)(\varphi_3+\psi_3)B_{13} + (\beta-1)\varphi_2 B_{12} \\ (\beta-\alpha)\psi_3 B_{13} + (\beta-1)\psi_2 B_{12} \end{pmatrix} \tag{3.12}$$

Further, letting $B = \begin{pmatrix} r \\ s \end{pmatrix} B_{23}$ we see that

$$\Sigma B(X,[Y,Z]) + \theta(X)B(Y,Z) = \Sigma \begin{pmatrix} (\beta-\alpha-1)r+s \\ (\beta-\alpha-1)s \end{pmatrix} x_1(y_2z_3-y_3z_2)$$

Hence B is a cocycle iff $\beta = \alpha+1$, in which case we obtain the following family of $\mathcal{N}$-admissible extensions

$$\mathcal{G}_{5,18}(\alpha) = \mathcal{G}\left(\begin{pmatrix} B_{23} \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha+1 & 1 \\ 0 & \alpha+1 \end{pmatrix}\right), \quad |\alpha| \geq 1 .$$

<u>3.2.4</u>. If $\theta(e_1) = \begin{pmatrix} \beta & \gamma \\ -\gamma & \beta \end{pmatrix}$, $\gamma \neq 0$, one sees easily that $H^2(\mathcal{G},\theta) = (0)$ for all β,γ.

Finally, if $\theta = 0$, then $H^2(\mathcal{G},\mathfrak{A}) = \left(\begin{pmatrix} B_{23} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ B_{23} \end{pmatrix}\right)$, so $H^2(\mathcal{G},R) = (B_{23})$ contains no two-dimensional subspaces $(G_2H^2(\mathcal{G},R) = \emptyset)$, and there is no admissible central extension of $\mathcal{G}_{3,2}(\alpha)$.

<u>3.3. Extensions of $\mathfrak{g}_{3,3}$</u>. The defining basis-relations of $\mathfrak{g} = \mathfrak{g}_{3,3}$ are $[e_1,e_2] = e_2+e_3$, $[e_1,e_3] = e_3$. Hence nilradical and commutator subalgebra both are equal to (e_2,e_3), and we let $\mathfrak{n} = (e_2,e_3)$. We discuss below all possible Jordan 2×2 normal forms $\theta(e_1)$. First we observe (cfr. [3, Corollary 2.6]),

<u>3.3.1. Lemma</u>. The automorphism group $\mathrm{Aut}(\mathfrak{g}_{3,3})$ is canonically isomorphic to the group of all matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ \varphi_1 & a & 0 \\ \varphi_2 & b & a \end{pmatrix}, \quad a \neq 0 .$$

From the above lemma it is clear that $\theta \circ A = \theta$ for all A in $\mathrm{Aut}(\mathfrak{g})$ and every representation $\theta : \mathfrak{g} \to \mathrm{End}(\mathbb{R}e_4 + \mathbb{R}e_5)$ with $\ker\theta = \mathfrak{n}$.

<u>3.3.2</u>. Assume $\theta(e_1) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, and let $f = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ be linear, as in 3.1, $f : \mathfrak{g} \to \mathfrak{a} = \mathbb{R}e_4 + \mathbb{R}e_5$. Calculating coboundaries we find,

$$df = \begin{pmatrix} ((\alpha-1)\varphi_2 - \varphi_3)B_{12} + (\alpha-1)\varphi_3 B_{13} \\ ((\beta-1)\psi_2 - \psi_3)B_{12} + (\beta-1)\psi_3 B_{13} \end{pmatrix} \tag{3.13}$$

Next, with $B = \begin{pmatrix} r \\ s \end{pmatrix} B_{23}$, we have

$$\begin{aligned} &\Sigma B(X,[Y,Z]) + \theta(X)B(Y,Z) \\ &= \Sigma \begin{pmatrix} (2-\alpha)r \\ (2-\beta)s \end{pmatrix} x_2(y_1z_3 - y_3z_1) \end{aligned} \tag{3.14}$$

Hence B is a cocycle iff $(2-\alpha)r = 0$ and $(2-\beta)s = 0$. Moreover, one sees that $\begin{pmatrix} t \\ u \end{pmatrix} B_{13}$ satisfies the cocycle identity for all α, β and all t, u. From this fact, (3.13) and (3.14) we can find $H^2(\mathfrak{g}, \theta)$.

Next we wish to apply Theorem 2. Observe that by Lemma 3.3.1 the isotropy-group $K(\theta)$ under $\mathrm{Aut}\,\mathfrak{g} \times \mathrm{Aut}\,\mathfrak{a}$ contains $\mathrm{Aut}\,\mathfrak{g}$, and furthermore, one finds $\psi\theta(e_1)\psi^{-1} = \theta(e_1)$ iff $\psi = \mathrm{diag}(x,y)$, for some (x,y) with $xy \neq 0$, (here $\psi \in \mathrm{Aut}\,\mathfrak{a} = GL(2,\mathbb{R})$). Thus $K(\theta)$ consists of all invertible 2×2 diagonal matrices. Let $A \in \mathrm{Aut}(\mathfrak{g})$. Then

$$A^t B_{23} A = a^2 B_{23} \quad \mod B^2(\mathfrak{g},\theta) \tag{3.15}$$

$$A^t B_{13} A = a\, B_{13} \quad \mod B^2(\mathfrak{g},\theta) \tag{3.16}$$

Put $\psi = \mathrm{diag}(x,y)$. Then, for $B = \begin{pmatrix} a^2 B_{23} \\ a\, B_{13} \end{pmatrix}$,

$$\psi \circ B = \begin{pmatrix} xa^2 & B_{23} \\ ya & B_{13} \end{pmatrix}, \quad xav \neq 0, \tag{3.17}$$

and

$$\psi\left(A^t \begin{pmatrix} B_{23} \\ 0 \end{pmatrix} A\right)\psi^{-1} = \begin{pmatrix} xa^2 & B_{23} \\ & 0 \end{pmatrix}, \quad \mod B^2(\mathfrak{g},\theta) \tag{3.18}$$

We are now ready to discuss the various cases of $H^2(\mathfrak{g},\theta)$, (cfr. (3.13) and (3.14)).

<u>$\alpha = \beta = 2$:</u> $H^2(\mathfrak{g},\theta) = \left(\begin{pmatrix} B_{23} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ B_{23} \end{pmatrix}\right)$. By (3.15) and (3.17) $K(\theta)$ acts transitively on $H^2(\mathfrak{g},\theta)$. Hence there is one admissible extension (within isomorphisms),

$$\mathfrak{g}_{5,19}(2) = \left(\begin{pmatrix} B_{23} \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right) \tag{3.19}$$

<u>$\alpha = 2,\ \beta = 1$:</u> $H^2(\mathfrak{g},\theta) = \left(\begin{pmatrix} B_{23} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ B_{13} \end{pmatrix}\right)$. By (3.17) and (3.18) the orbit space $H^2(\mathfrak{g},\theta)/K(\theta)$ consists of exactly two orbits, passing through $\begin{pmatrix} B_{23} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} B_{23} \\ B_{13} \end{pmatrix}$ respectively. This gives the two admissible extensions

$$\mathfrak{g}\left(\begin{bmatrix} B_{23} \\ 0 \end{bmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\right) \quad \text{and} \quad \mathfrak{g}\left(\begin{bmatrix} B_{23} \\ B_{13} \end{bmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\right) \tag{3.20}$$

<u>$\alpha = 1,\ \beta = 2$:</u> $H^2(\mathfrak{g},\theta) = \left(\begin{bmatrix} B_{13} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ B_{23} \end{bmatrix}\right)$. This case is equivalent to the previous case by symmetry.

<u>$\alpha = 2,\ \beta \neq 1,\ \beta \neq 2$:</u> $H^2(\mathfrak{g},\theta) = \left(\begin{bmatrix} B_{23} \\ 0 \end{bmatrix}\right)$. Obviously there is exactly one $K(\theta)$-orbit, giving the extension $\mathfrak{g}\left(\begin{bmatrix} B_{23} \\ 0 \end{bmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & \end{pmatrix}\right)$.

<u>$\alpha \neq 1,\ \alpha \neq 2,\ \beta = 2$:</u> This case gives an extension isomorphic to the previous one by symmetry (let $\psi = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$). To sum up we find the following admissible extensions:

$$\mathfrak{g}_{5,19}(\alpha) = \left(\begin{bmatrix} B_{23} \\ 0 \end{bmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & \beta \end{pmatrix}\right), \quad \alpha \neq 0 \tag{3.21}$$

$$\mathfrak{g}_{5,20} = \left(\begin{bmatrix} B_{23} \\ B_{13} \end{bmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\right) \tag{3.22}$$

<u>3.3.3</u>. Assume $\theta(e_1) = \left(\begin{smallmatrix} \alpha & 1 \\ 0 & \alpha \end{smallmatrix}\right)$. Calculating coboundaries we find,

$$B^2(\mathfrak{g},\theta) = \left(\begin{bmatrix} B_{12} \\ 0 \end{bmatrix}, \begin{bmatrix} B_{13} \\ -B_{13} \end{bmatrix}\right), \text{ when } \alpha = 1 \tag{3.23}$$

and

$$B^2(\mathfrak{g},\theta) = \left(\begin{bmatrix} B_{12} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ B_{12} \end{bmatrix}, \begin{bmatrix} B_{13} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ B_{13} \end{bmatrix}\right), \text{ when } \alpha \neq 1 \tag{3.24}$$

Moreover, $\left(\begin{smallmatrix} r \\ s \end{smallmatrix}\right)B_{23}$ is a cocycle iff $\alpha = 2$ and $s = 0$, and in this case $H^2(\mathfrak{g},\left(\begin{smallmatrix} 2 & 1 \\ 0 & 2 \end{smallmatrix}\right)) = \left(\begin{bmatrix} B_{23} \\ 0 \end{bmatrix}\right)$.

It follows easily that the only admissible extension is

$$\mathfrak{g}\left(\begin{bmatrix} B_{23} \\ 0 \end{bmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}\right) = \mathfrak{g}_{5,21} \tag{3.25}$$

<u>3.3.4</u>. Assume $\theta(e_1) = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, $\beta \neq 0$. The coboundaries are

$$B^2(\mathfrak{g},\theta) = (\begin{pmatrix} B_{12} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ B_{12} \end{pmatrix}, \begin{pmatrix} B_{13} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ B_{13} \end{pmatrix}) \tag{3.26}$$

Further, for X,Y,Z in $\mathfrak{g}$,

$$\begin{aligned} &\Sigma\binom{r}{s}B_{23}(X,[Y,Z]) + \theta(X)\binom{r}{s}B_{23}(Y,Z) \\ &= \Sigma x_1 \begin{pmatrix} (\alpha-2)r+\beta s \\ -\beta r+(\alpha-2)s \end{pmatrix} B_{23}(Y,Z) \end{aligned} \tag{3.27}$$

From (3.26) and (3.27) it follows immediately that $H^2(\mathfrak{g},\theta) = (0)$ for all θ, and there is no admissible extension.

Finally, if $\theta = 0$, B_{12} and B_{13} are coboundaries, and B_{23} is no cocycle. Hence there are no admissible central extensions of $\mathfrak{g}_{3,3}$.

<u>3.4. Extensions of $\mathfrak{g}_{3,4}(\alpha)$.</u> $\mathfrak{g} = \mathfrak{g}_{3,4}(\alpha)$, $\alpha \geq 0$, is defined by the basis relations $[e_1,e_2] = \alpha e_2 - e_3$, $[e_1,e_3] = e_2 + \alpha e_3$. Hence nilradical and commutator subalgebra are both equal to $\mathbb{R}e_2 + \mathbb{R}e_3$, and is abelian. Hence we let $\mathfrak{n} = [\mathfrak{g},\mathfrak{g}]$. We shall need the following

<u>3.4.1. Lemma.</u> The automorphism group $\mathrm{Aut}(\mathfrak{g}_{3,4}(\alpha))$, $\alpha \geq 0$, is isomorphic to the group of all real matrices

(a) $\qquad A_+ = \begin{pmatrix} 1 & 0 & 0 \\ \varphi_1 & a & b \\ \varphi_2 & -b & a \end{pmatrix}$, $a^2+b^2 \neq 0$, whenever $\alpha > 0$

(b) $\qquad A_- = \begin{pmatrix} -1 & 0 & 0 \\ \varphi_1 & a & b \\ \varphi_2 & b & -a \end{pmatrix}$, $a^2+b^2 \neq 0$, and A_+ (as in (a)), whenever $\alpha = 0$.

<u>Proof</u>. $\mathfrak{g}_{3,4}(\alpha)$ may be viewed as an extension of Re_1 by $\mathfrak{a} = Re_2 + Re_3$ (both abelian Lie algebras). Here $\mathfrak{a}$ is the center of the extended Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{g}_{3,4}(\alpha)$. Hence by [3, Corollary 2.6], any A in $\mathrm{Aut}\,\mathfrak{g}_{3,4}(\alpha)$ has a matrix representation

$$A = \begin{pmatrix} \alpha_0 & 0 & 0 \\ \varphi_1 & a & b \\ \varphi_2 & c & d \end{pmatrix}, \quad \alpha_0 \neq 0 , \quad ad-bc \neq 0 .$$

Letting $A[X,Y] = [AX,AY]$, for all $X,Y \in \mathfrak{g}$, we obtain the following system of equations

$$\left.\begin{array}{rcl} \alpha(\alpha_0-1)a + b + \alpha_0 c & = & 0 \\ -a + \alpha(\alpha_0-1)b + \alpha_0 d & = & 0 \\ -\alpha_0 a + \alpha(\alpha_0-1)c + d & = & 0 \\ -\alpha_0 b - c + \alpha(\alpha_0-1)d & = & 0 \end{array}\right\} \tag{3.28}$$

Now the determinant of (3.28) equals

$$p(\alpha_0) = (\alpha_0-1)[4\alpha^2\alpha_0^3 - 4\alpha^2\alpha_0^2+\alpha_0+(1-\alpha^2)] \tag{3.29}$$

where the polynomial $p(\alpha_0)$ has at most four real zeros. Moreover, the α_0-s must form a subgroup of $\mathrm{Aut}(\mathfrak{g}_1)$, i.e., of the multiplicative group of reals. Hence the only possibilities are $\alpha_0 \pm 1$. Now if $\alpha = 0$ we obviously have the solutions $\alpha_0 = \pm 1$. If $\alpha > 0$ only $\alpha_0 = 1$ is possible since substitution of $\alpha_0 = -1$ in (3.29) gives $\alpha = 0$. After these remarks the system (3.28) can easily be solved for a, b, and c, and the lemma follows.

Next we wish to apply Theorem 2. Thus let $\theta : \mathfrak{g} \to \mathrm{End}\,\mathfrak{a}$ be a representation with $\ker\theta = \mathfrak{n}$, $\mathfrak{a} = Re_4+Re_5 \approx (\mathfrak{g}_1)^2$. In this case θ is uniquely determined by the single endomorphism $\theta(e_1)$. Therefore we consider the set of all pairwise non-equivalent representations θ for which $\theta(e_1)$ is a 2×2 Jordan matrix relative to the basis

$\{e_4,e_5\}$ for $\mathfrak{A}$. That is, $\theta(e_1)$ has one of the following forms: $\begin{pmatrix}\beta & 0\\ 0 & \gamma\end{pmatrix}$, $\begin{pmatrix}\beta & 1\\ 0 & \beta\end{pmatrix}$, $\begin{pmatrix}\beta & \gamma\\ -\gamma & \beta\end{pmatrix}$.

<u>3.1.2.</u> Let $\theta(e_1) = \begin{pmatrix}\beta & 0\\ 0 & \gamma\end{pmatrix}$, $\theta(e_2) = \theta(e_3) = 0$. Assume $\mathfrak{A} = Re_4 \oplus Re_5$. We calculate $B^2(\mathfrak{g},\theta)$. Therefore let $f = \begin{pmatrix}\varphi\\ \psi\end{pmatrix}: \mathfrak{g} \to \mathfrak{A}$ be linear (coordinates relative to $\{e_4,e_5\}$). Then for X,Y in $\mathfrak{g}$,

$$B_f(X,Y) = \theta(X)f(Y) - \theta(Y)f(X) - f[X,Y]$$
$$= \begin{pmatrix}(\beta\varphi_2-\alpha\varphi_2+\varphi_3)B_{12} + (\beta\varphi_3-\varphi_2-\alpha\varphi_3)B_{13}\\ (\gamma\psi_2-\alpha\psi_2+\psi_3)B_{12} + (\gamma\psi_3-\psi_2-\alpha\psi_3)B_{13}\end{pmatrix} (X,Y)$$

where

$$\varphi = (\varphi_1,\varphi_2,\varphi_3) \text{ and } \psi = (\psi_1,\psi_2,\psi_3) \text{ relative to } \varepsilon^* .$$

Hence

$$B^2(\mathfrak{g},\theta) = \left(\begin{pmatrix}B_{12}\\ 0\end{pmatrix}, \begin{pmatrix}0\\ B_{12}\end{pmatrix}, \begin{pmatrix}B_{13}\\ 0\end{pmatrix}, \begin{pmatrix}0\\ B_{13}\end{pmatrix}\right),$$

relative to $\{e_4,e_5\}$. It remains to check if $\begin{pmatrix}B_{23}\\ 0\end{pmatrix}$ and $\begin{pmatrix}0\\ B_{23}\end{pmatrix}$ generate any cocycles. Computations give

$$\Sigma B_{23}(X,[Y,Z]) = 2\alpha\, \Sigma x_2(y_1z_3-y_3z_1)$$

and

$$\Sigma\, \theta(X)\begin{pmatrix}rB_{23}\\ sB_{23}\end{pmatrix}(Y,Z) = \Sigma\begin{pmatrix}\beta r\\ \gamma s\end{pmatrix}(y_2z_3-y_3z_2)$$

Hence

$$\Sigma \begin{pmatrix}r\\ s\end{pmatrix}B_{23}(X,[Y,Z]) - \theta(X)\begin{pmatrix}r\\ s\end{pmatrix}B_{23}(Y,Z) = 0 , \text{ all } X,Y,Z$$

if and only if

$$\left.\begin{aligned}(2\alpha-\beta)r &= 0\\ (2\alpha-\gamma)s &= 0\end{aligned}\right\} \tag{3.30}$$

Solving the system (3.30) we obtain,

$$H^2(\mathcal{G},\theta) = \begin{cases} B_{23}Re_4 , & \text{whenever } \theta(e_1) = \begin{pmatrix} 2\alpha & 0 \\ 0 & \gamma \end{pmatrix} , \ \gamma \neq 2\alpha \\ B_{23}(Re_4+Re_5), & \text{whenever } \theta(e_1) = 2\alpha\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \alpha > 0 \end{cases}$$

In addition, $H^2(\mathcal{G},\theta) = B_{23}Re_5$ if $\theta(e_1) = \begin{pmatrix} \beta & 0 \\ 0 & 2\alpha \end{pmatrix}$, $\beta \neq 2\alpha$. But this case can be omitted because of symmetry.

Further it is easily seen that

$$(A_+)^t B_{23} A_+ = (a^2+b^2)B_{23} + (a\varphi_1 - b\varphi_2)B_{12} + (b\varphi_1 - a\varphi_2)B_{13}$$

and

$$(A_-)^t B_{23} A_- = -(a^2+b^2)B_{23} + (b\varphi_1 - a\varphi_2)B_{12} - (a\varphi_1 + b\varphi_2)B_{13}$$

Now for $\theta(e_1) = \begin{pmatrix} 2\alpha & 0 \\ 0 & \gamma \end{pmatrix}$, $\gamma \neq 2\alpha$, the fixed point group $K(\theta)$ of θ in $\operatorname{Aut}\mathcal{G} \times \operatorname{Aut}\mathfrak{A}$ is given as follows:

$$K(\theta) = K(\theta)^+ = \{(A_+,\psi) : \psi = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \text{ or } \psi = \begin{pmatrix} 0 & y \\ y^{-1} & 0 \end{pmatrix}, x \neq 0, y \neq 0\} , \text{ if } \alpha > 0,$$

and

$$K(\theta) = \{(A_-,\begin{pmatrix} -x & 0 \\ 0 & x^{-1} \end{pmatrix}) : x \neq 0\} \cup K(\theta)^+ , \quad \text{if } \alpha = 0 .$$

It follows that $H(\mathcal{G},\theta)$ consists of exactly one $K(\theta)$-orbit for every θ .

<u>3.4.3. Proposition</u>. The admissible extensions of $\mathcal{G}_{3,4}(\alpha)$, $\alpha \geq 0$, bys $\mathfrak{A} = Re_4 + Re_5$ corresponding to the representation θ with $\theta(e_1) = \begin{pmatrix} 2\alpha & 0 \\ 0 & \beta \end{pmatrix}$, $\alpha > 0$ or $\beta \neq 0$, are within isomorphisms the Lie algebras

$$(\begin{bmatrix} B_{23} \\ 0 \end{bmatrix},\theta) = \mathcal{G}_{5,22}(\alpha,\beta) .$$

These Lie algebras are pairwise non-isomorphic.

3.4.4. Let $\theta(e_1) = \begin{pmatrix} \beta & 1 \\ 0 & \beta \end{pmatrix}$, $\ker\theta = Re_2 + Re_3 = \mathfrak{n}$. Let us calculate $B^2(\mathfrak{g},\theta)$. If $f = \begin{pmatrix} \varphi \\ \psi \end{pmatrix} : \mathfrak{g}_{3,4}(\alpha) \to \mathfrak{a}$ is linear, $\mathfrak{a} = Re_4 + Re_5$, then if $\varphi = (\varphi_1,\varphi_2,\varphi_3)$, $\psi = (\psi_1,\psi_2,\psi_3)$,

$$B_f(X,Y) = \theta(X)f(Y) - \theta(Y)f(X) - f[X,Y]$$

$$= \begin{bmatrix} (\varphi_2+\psi_2-\alpha\varphi_2+\varphi_3)B_{12} + (\varphi_3+\psi_3-\varphi_2-\alpha\varphi_3)B_{13} \\ (\psi_2-\alpha\varphi_2+\varphi_3)B_{12} + (\psi_3-\varphi_2-\alpha\varphi_3)B_{13} \end{bmatrix}(X,Y), \quad X,Y \in \mathfrak{g}.$$

Hence

$$B^2(\mathfrak{g},\theta) = (B_{12}e_4, B_{12}e_5, B_{13}e_4, B_{13}e_5)$$

Next, let $B = \begin{pmatrix} r \\ s \end{pmatrix} B_{23}$. We check the cocycle identity,

$$\Sigma(\theta(X)B(Y,Z) + B(X,[Y,Z]))$$

$$= \Sigma(x_1\begin{pmatrix} \beta r+s \\ \beta s \end{pmatrix}B_{23}(Y,Z) - x_1 2\alpha\begin{pmatrix} r \\ s \end{pmatrix}B_{23}(Y,Z))$$

Hence B is a cocycle if and only if

$$(\beta-2\alpha)r + s = 0$$

$$(\beta-2\alpha)s = 0$$

Solving this system we obtain,

$$H^2(\mathfrak{g},\theta) = \begin{cases} B_{23}Re_4, & \beta = 2\alpha \\ (0), & \beta \neq 2\alpha \end{cases}$$

Now the fixed point group of θ is seen to be,

$$K(\theta) = (\mathrm{Aut}\,\mathfrak{g})_0 \times \{\begin{pmatrix} \pm 1 & y \\ 0 & v \end{pmatrix} : v \neq 0\}, \quad \alpha \geq 0,$$

where $(\mathrm{Aut}\,\mathfrak{g})_0$ is the identity component of $\mathrm{Aut}\,\mathfrak{g}$ ($(\mathrm{Aut}\,\mathfrak{g})_0 = \mathrm{Aut}\,\mathfrak{g}$ unless $\alpha = 0$). Since $(A_+)^t B_{23} A_+ = (a^2+b^2)B_{23}$, mod $B^2(\mathfrak{g},\theta)$, it follows that $H^2(\mathfrak{g},\theta)$ consist of exactly one $K(\theta)$-orbit.

3.4.5. Proposition. The extensions of $\mathcal{G}_{3,4}(\alpha)$ by $\mathfrak{a}$ corresponding to the representation $\begin{pmatrix}\beta & 1\\ 0 & \beta\end{pmatrix}$ $(=\theta(e_1))$ are, within idomorphisms, the Lie algebras

$$\mathcal{G}_{5,23}(\alpha) = \mathcal{G}\left(\begin{pmatrix}B_{23}\\ 0\end{pmatrix}, \begin{pmatrix}2\alpha & 1\\ 0 & 2\alpha\end{pmatrix}\right).$$

These Lie algebras are pairwise non-isomorphic.

3.4.6. Let $\theta(e_1) = \begin{pmatrix}\beta & \gamma\\ -\gamma & \beta\end{pmatrix}$. If $f = \begin{pmatrix}\varphi\\ \psi\end{pmatrix} : \mathcal{G} \to \mathfrak{a}$ is linear, $\varphi = (\varphi_1, \varphi_2, \varphi_3)$, $\psi = (\psi_1, \psi_2, \psi_3)$, then

$$\theta(X)f(Y) - \theta(Y)f(X) - f[X,Y]$$
$$= \begin{pmatrix}(\beta\varphi_2+\psi_2-\alpha\varphi_2+\varphi_3)B_{12} + (\beta\varphi_3+\psi_3-\varphi_2-\alpha\varphi_3)B_{13}\\ (-\varphi_2+\beta\psi_2-\alpha\psi_2+\psi_3)B_{12} + (-\varphi_3+\beta\psi_3-\psi_2-\alpha\psi_3)B_{13}\end{pmatrix}(X,Y)$$

and $\dim B^2(\mathcal{G},\theta) = 4$.

Checking the cocycle identity for $B = \begin{pmatrix}r\\ s\end{pmatrix}B_{23}$, we end up with the system

$$(2\alpha-\beta)r - s = 0$$
$$r + (2\alpha-\beta)s = 0$$

which has a determinant $(2\alpha-\beta)^2+1 \neq 0$, hence has only the trivial solution $r = s = 0$. This gives $H^2(\mathcal{G},\theta) = (0)$ and there are no extensions. This completes our discussion of $\mathcal{G}_{3,4}(\alpha)$.

3.5. Extensions of $(\mathcal{G}_1)^3$. Let $\mathcal{G} = (\mathcal{G}_1)^3$ with a basis $(e_i)_{i=1}^3$. Since $[\mathcal{G},\mathcal{G}] = (0)$ and $\mathcal{G}$ is nilpotent, there are several cases for nilpotent subalgebra $\mathcal{N}$. Let θ be a representation of $\mathcal{G}$ in $\mathfrak{a} = Re_4 + Re_5$. For each $\mathcal{N}$ we assume $\ker\theta \supset \mathcal{N}$.
Now if $\mathcal{N} = \mathcal{G}$ then $\theta = 0$, and all ($\mathcal{N}$-admissible) central extensions of $\mathcal{G}$ by $\mathfrak{a}$ are known to be isomorphic with $\mathcal{N}_{5,2} = \mathcal{G}\left(\begin{pmatrix}B_{12}\\ B_{13}\end{pmatrix}\right)$.

3.5.1. Next, if $\dim \mathfrak{N} = 2$ we may clearly assume $\mathfrak{N} = (e_2, e_3)$, and θ is uniquely given by the single endomorphism $\theta(e_1)$, which we may assume is represented by a Jordan normal form. Hence we discuss the following cases.

Let $\theta(e_1) = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$. The form $B = \begin{pmatrix} r \\ s \end{pmatrix} B_{23}$ is a cocycle iff $\alpha = 0$ and $s = 0$. Obviously $\mathfrak{z}_{B^0} \cap \mathfrak{Z}(\mathfrak{N}) = (0)$, however the extension $\mathfrak{g}(\begin{pmatrix} B_{23} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})$ is isomorphic with the direct product $\mathfrak{g}_{3,1} \times \mathfrak{g}_2$ (Lie relations $[e_1, e_5] = e_5$, $[e_2, e_3] = e_4$). This is seen to be the only $\mathfrak{N}$-admissible extension $(B^2(\mathfrak{g}, \theta) = (\begin{pmatrix} 0 \\ B_{12} \end{pmatrix}, \begin{pmatrix} 0 \\ B_{13} \end{pmatrix}))$. Next, if $\theta(e_1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\theta(e_1)$ is nilpotent, and this case is omitted, cfr. (1.2).

Let $\theta(e_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then one has $B^2(\mathfrak{g}, \theta) = (\begin{pmatrix} B_{12} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ B_{12} \end{pmatrix}, \begin{pmatrix} B_{13} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ B_{13} \end{pmatrix})$, moreover $\begin{pmatrix} r \\ s \end{pmatrix} B_{23}$ is no cocycle. Hence we find no extensions.

Finally let $\theta(e_1) = \begin{pmatrix} \beta & 1 \\ -1 & \beta \end{pmatrix}$. Again $\begin{pmatrix} r \\ s \end{pmatrix} B_{23}$ is no cocycle, and $B^2(\mathfrak{g}, \theta)$ is as in the previous case.

Remark. Since $\mathrm{Aut}\, \mathfrak{g} = GL(3, R)$ it is seen that $a\,\theta(e_1) = \theta(Ae_1)$, and $A(\mathfrak{N}) = \mathfrak{N}$, where $A = \mathrm{diag}(a, a^{-1}, 1)$, $a \neq 0$. Hence it suffices to consider $\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$ rather than $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, etc. ... , above.

3.5.2. If $\dim \mathfrak{N} = 1$, then $\mathfrak{z}_{B^0} \cap \mathfrak{Z}(\mathfrak{N}) \neq (0)$ for all alternating bilinear maps B. Hence this case gives no new Lie algebra. Next assume $\mathfrak{N} = (0)$. Then $\theta(e_1), \theta(e_2)$, and $\theta(e_3)$ commute. Clearly we may assume $\theta(e_1)$ is a Jordan normal form.

Let $\theta(e_1) = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$, and assume $\theta(e_i) \neq \lambda I$ for all λ and all $i = 1,2,3$. Then we must have $\theta(e_2) = \mathrm{diag}(\beta, \gamma)$, $\theta(e_3) = (\lambda, \mu)$, and since $(\alpha, 1)$, (β, λ), (λ, μ) are linearly dependent, $\ker\theta \neq (0)$. Therefore there are no $\mathfrak{N}$-admissible extensions, see (1.2)

Assume next $\theta(e_1) = I$. Then $\theta(e_2)$ and $\theta(e_3)$ can be any commuting 2×2-matrices, and we may assume $\theta(e_2)$ is a Jordan normal form. Now, if $\theta(e_2)$ is a diagonal matrix then so is $\theta(e_3)$, and we find $\ker\theta \neq (0)$ as above.

Suppose $\theta(e_2) = \begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix}$, hence $\theta(e_3) = \begin{pmatrix}\beta & \gamma\\ 0 & \beta\end{pmatrix}$, and again $\ker\theta \neq (0)$.

Finally, let $\theta(e_2) = \begin{pmatrix}\alpha & 1\\ -1 & \alpha\end{pmatrix}$, $\theta(e_3) = \begin{pmatrix}\beta & \gamma\\ -\gamma & \beta\end{pmatrix}$. It is immediate that $\ker\theta \neq (0)$.

Similar arguments apply when $\theta(e_1) = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, and $\theta(e_1) = \begin{pmatrix}\alpha & 1\\ -1 & \alpha\end{pmatrix}$.

Hence there are no $\mathfrak{N}$-admissible extensions.

This completes our discussion of the case $\mathfrak{N} = (0)$.

<u>3.6. Extensions of $\mathfrak{g}_2 \times \mathfrak{g}_1$</u>. We define $\mathfrak{g} = \mathfrak{g}_2 \times \mathfrak{g}_1$ by the basis relation $[e_1,e_2] = e_2$, e_3 is central. In this case $[\mathfrak{g},\mathfrak{g}] = (e_2)$ whereas the nilradical is (e_2,e_3). If $\mathfrak{N} = (e_2)$ no cocycle B satisfies $\mathfrak{s}_{B^0} \cap \mathfrak{z}(\mathfrak{N}) = (0)$ and all semidirect products are seen to be direct products, hence we put $\mathfrak{N} = (e_2,e_3)$.

Let $\theta : \mathfrak{g} \to \mathrm{End}(Re_4+Re_5)$ be a representation with $\ker\theta = \mathfrak{N}$. As before it suffices to discuss the various Jordan normal forms $\theta(e_1)$.

<u>3.6.1. Lemma</u>. The automorphism group $\mathrm{Aut}(\mathfrak{g}_2\times\mathfrak{g}_1)$ is canonically isomorphic with the group of all matrices

$$A = \begin{pmatrix}1 & 0 & 0\\ a & b & 0\\ c & d & e\end{pmatrix}, \quad e \neq 0,\ b \neq 0. \tag{3.31}$$

The above lemma implies that $\theta\circ A = \theta$ for every A in $\mathrm{Aut}(\mathfrak{g})$ and all representations θ with $\ker\theta = \mathfrak{N}$.

<u>3.6.2.</u> Let $\theta(e_1) = \begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$, and let $f = \begin{pmatrix}\varphi\\ \psi\end{pmatrix} : \mathfrak{g} \to \mathfrak{a} = Re_4 + Re_5$ be linear, $\varphi = \sum_{i=1}^{3}\varphi_i e_i^*$, $\psi = \sum_{i=1}^{3}\psi_i e_i^*$. Then $df = (\alpha-\beta)\begin{pmatrix}\varphi_2\\ \psi_2\end{pmatrix}B_{12}$,

hence we find

$$B^2(\mathcal{G},\theta) = \left(\begin{pmatrix}B_{12}\\0\end{pmatrix},\begin{pmatrix}0\\B_{12}\end{pmatrix}\right) , \quad \alpha \neq \beta \tag{3.32}$$

$$B^2(\mathcal{G},\theta) = (0) \qquad , \quad \alpha = \beta \tag{3.33}$$

Checking the cocycle identity we obtain

$$H^2(\mathcal{G},\theta) = \left(\begin{pmatrix}B_{23}\\0\end{pmatrix},\begin{pmatrix}0\\B_{23}\end{pmatrix},\begin{pmatrix}B_{12}\\0\end{pmatrix},\begin{pmatrix}0\\B_{12}\end{pmatrix},\begin{pmatrix}B_{13}\\0\end{pmatrix},\begin{pmatrix}0\\B_{13}\end{pmatrix}\right), \quad \alpha=\beta=1 \tag{3.34}$$

$$H^2(\mathcal{G},\theta) = \left(\begin{pmatrix}B_{23}\\0\end{pmatrix},\begin{pmatrix}B_{13}\\0\end{pmatrix},\begin{pmatrix}0\\B_{13}\end{pmatrix}\right), \quad \alpha = 1,\ \beta \neq 1 \tag{3.35}$$

$$H\ (\mathcal{G},\theta) = \left(\begin{pmatrix}0\\B_{23}\end{pmatrix},\begin{pmatrix}B_{13}\\0\end{pmatrix},\begin{pmatrix}0\\B_{13}\end{pmatrix}\right), \quad \alpha \neq 1,\ \beta = 1 \tag{3.36}$$

Moreover, if $\alpha \neq 1$ and $\beta \neq 1$, $\mathcal{S}_{B^0} \cap \mathcal{Z}(\mathcal{N}) \neq (0)$ for all B in $H^2(\mathcal{G},\theta)$. We observe also that (3.36) is transformed into (3.35) by $\psi = \begin{pmatrix}0&1\\1&0\end{pmatrix} \in \mathrm{Aut}\,\mathfrak{A}$.

Using Lemma 3.6.1 we find

$$A^tB_{12}A = bB_{12} \ , \quad A^tB_{13}A = dB_{12} + eB_{13} \tag{3.37}$$

$$A^tB_{23}A = (ad-bc)B_{12} + acB_{13} + beB_{23} \tag{3.38}$$

Now if $\alpha = \beta = 1$ (as in (3.34)), then $\theta(e_1) = I$ is left fixed by all of $\mathrm{Aut}(\mathfrak{A})$. Hence $K(\theta) = \mathrm{Aut}(\mathcal{G}) \times \mathrm{Aut}(\mathfrak{A})$. Combining this with (3.37) and (3.38) the orbit space $H^2(\mathcal{G},\theta)/K(\theta)$ is seen to consist of exactly three orbits, passing through the cocycles $\begin{pmatrix}B_{23}\\0\end{pmatrix}$, $\begin{pmatrix}B_{23}\\B_{12}\end{pmatrix}$, $\begin{pmatrix}B_{23}\\B_{12}\end{pmatrix}$, respectively. Further, if $\alpha \neq 1$ and $\beta = 1$ (as in (3.35)), there are two $K(\theta)$-orbits in $H^2(\mathcal{G},\theta)$ for each value of α. These orbits are determined by the cocycles $\begin{pmatrix}B_{13}\\B_{23}\end{pmatrix}$ and $\begin{pmatrix}0\\B_{23}\end{pmatrix}$, respectively.

Combining this we obtain the following three families of $\mathfrak{n}$-admissible extensions,

$$\mathfrak{g}_{5,10}(\alpha) = \mathfrak{g}\left(\begin{pmatrix} 0 \\ B_{23} \end{pmatrix}, \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right) \tag{3.39}$$

$$\mathfrak{g}_{5,11}(\alpha) = \mathfrak{g}\left(\begin{pmatrix} B_{13} \\ B_{23} \end{pmatrix}, \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right) \tag{3.40}$$

$$\mathfrak{g}_{5,12} = \mathfrak{g}\left(\begin{pmatrix} B_{23} \\ B_{12} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \tag{3.41}$$

The above Lie algebras are pairwise non-isomorphic.

3.6.3. Let $\theta(e_1) = \left(\begin{smallmatrix} \alpha & 1 \\ 0 & \alpha \end{smallmatrix}\right)$. Then one finds

$$B^2(\mathfrak{g},\theta) = \left(\begin{pmatrix} B_{12} \\ 0 \end{pmatrix}, \begin{pmatrix} B_{13} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ B_{13} \end{pmatrix}\right), \quad \alpha = 1 \tag{3.42}$$

$$B^2(\mathfrak{g},\theta) = \left(\begin{pmatrix} B_{12} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ B_{12} \end{pmatrix}, \begin{pmatrix} B_{13} \\ 0 \end{pmatrix}\right), \quad \alpha = 0 \tag{3.43}$$

$$B^2(\mathfrak{g},\theta) = \left(\begin{pmatrix} B_{12} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ B_{12} \end{pmatrix}, \begin{pmatrix} B_{13} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ B_{13} \end{pmatrix}\right), \quad \alpha \neq 0 \text{ and } \alpha \neq 1 \tag{3.44}$$

Furthermore, checking the cocycle identity we obtain

$$H^2(\mathfrak{g},\theta) = \left(\begin{pmatrix} B_{23} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ B_{12} \end{pmatrix}\right), \quad \alpha = 1 \tag{3.45}$$

$$H^2(\mathfrak{g},\theta) = \left(\begin{pmatrix} 0 \\ B_{13} \end{pmatrix}\right), \quad \alpha = 0 \tag{3.46}$$

$$H^2(\mathfrak{g},\theta) = (0), \quad \alpha \neq 0 \text{ and } \alpha \neq 1 \tag{3.47}$$

Only the case $\alpha = 1$ can give $\mathfrak{n}$-admissible extensions, and in this case the isotropy group $K(\theta)$ of θ in $\operatorname{Aut}\mathfrak{g} \times \operatorname{Aut}\mathfrak{a}$ is $\operatorname{Aut}\mathfrak{g} \times H$, where H consists of all matices $\psi = \left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$, $x \neq 0$. Combining this with (3.37) and (3.38) we obtain two $K(\theta)$-orbits in

$H^2(\mathfrak{g},\theta)$, and they yield the following $\mathfrak{n}$-admissible extensions,

$$\mathfrak{g}_{5,15} = \mathfrak{g}\left(\begin{bmatrix} B_{23} \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) \tag{3.48}$$

$$\mathfrak{g}_{5,16} = \mathfrak{g}\left(\begin{bmatrix} B_{23} \\ B_{12} \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) \tag{3.49}$$

<u>3.6.4</u>. Let $\theta(e_1) = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, $\beta \neq 0$. One finds $H^2(\mathfrak{g},\theta) = (0)$, all α, β .

Let $\theta = 0$. Since $\Sigma B_{23}(X,[Y,Z]) \neq 0$ we obtain no $\mathfrak{n}$-admissible central extensions.

This completes our discussion of $\mathfrak{g}_2 \times \mathfrak{g}_1$.

4. Extensions of $(\mathfrak{g}_1)^2$. Let $\mathfrak{g} = (\mathfrak{g}_1)^2$ (for $\mathfrak{g} = \mathfrak{g}_2$ see 4.8), $\mathfrak{a} = (\mathfrak{g}_1)^3$. Let $\theta : \mathfrak{g} \to \operatorname{End}\mathfrak{a}$ be a representation, and let $\mathfrak{n} \subset \mathfrak{g}$ be a subalgebra with $\ker\theta \supset \mathfrak{n}$. Now if $\mathfrak{n} = \mathfrak{g}$ then θ vanishes identically and we have only central extensions $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$ by $\mathfrak{a}$. Hence $\tilde{\mathfrak{g}}$ is nilpotent. Therefore we assume $\mathfrak{n} \neq \mathfrak{g}$. Let $\{e_1, e_2\}$ be a basis for $\mathfrak{g}$. If $\mathfrak{n}$ has dimension one then all bilinear cocycles $B : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{a}$ are degenerate on $\mathfrak{n}$. Hence we assume $\mathfrak{n} = (0)$.

Since $\mathfrak{g}$ is abelian we have $[\theta(X),\theta(Y)] = 0$ for all X, Y in $\mathfrak{g}$. Moreover, we need only consider those θ for which $\theta(X)$ is nilpotent if and only if $X = 0$ (Theorem 1). Further, we may always assume via a change of basis in $\mathfrak{a}$ that $\theta(e_1)$ is represented by a Jordan normal form. (For simplicity of notation we include matrices with complex eigenvalues such as $\begin{pmatrix}\alpha & 0 & 0\\ 0 & \beta & 1\\ 0 & -1 & \beta\end{pmatrix}$ among the Jordan forms.)

Now the non-nilpotent 3×3 Jordan forms are given (within a scalar factor) by the five following families of matrices A.

$$(1)\ \begin{pmatrix}0&0&0\\0&1&1\\0&0&1\end{pmatrix}\quad (2)\ \begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}\quad (3)\ \begin{pmatrix}1&0&0\\0&\alpha&1\\0&0&\alpha\end{pmatrix}\quad (4)\ \begin{pmatrix}1&0&0\\0&\alpha&0\\0&0&\beta\end{pmatrix}\quad (5)\ \begin{pmatrix}\alpha&0&0\\0&\beta&1\\0&-1&\beta\end{pmatrix}.$$

Next we determine all 3×3 matrices B that commute with the above operators A, and such that no nontrivial linear combination of A and B is nilpotent. Such a pair $\{A,B\}$ generates a representation θ of $\mathfrak{g}$ in $\mathfrak{a}$ (let $\theta(e_1) = A$, $\theta(e_2) = B$). Now $\operatorname{Aut}(\mathfrak{g}) = GL(2,R)$ operates on the pair $\{A,B\}$ in the following way, $\binom{a\ b}{c\ d}\{A,B\} = \{aA+bB,\ cA+dB\}$. We write $\{A,B\}\sim\{A_1,B_1\}$ in case the two pairs are in same $GL(2,R)$ orbit. Further, $\operatorname{Aut}(\mathfrak{a}) = GL(3,R)$ operates by $\Psi\{A,B\} = \{\Psi A\Psi^{-1}, \Psi A\Psi^{-1}\}$. By $\{A,B\} \overset{J}{\sim} \{A_1,B_1\}$ we mean that $\{A,B\}$ and $\{A_1,B_1\}$ are conjugate under $GL(3,R)$. By Theorem 2, classifying all irreducible semidirect products $\mathfrak{a} \times_\theta \mathfrak{g}$ amounts to determining all $GL(2,R) \times GL(3,R)$ orbits of commuting pairs $\{A,B\}$. It is to this task we now turn.

4.1. Let $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. The commutant of B consists of all matrices $B = \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & 0 & b \end{pmatrix}$. We assume $a \neq 0$, otherwise $bA - B$ is nilpotent.

4.1.1. Let $b \neq c$. Then

$$\{A,B\} \sim \{A-(c-b)^{-1}\, a(B-bA),\ a^{-1}(B-bA)\}$$

$$= \{\mathrm{diag}(1-a(c-b)^{-1},1,1),\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & a^{-1}(c-b) \\ 0 & 0 & 0 \end{pmatrix}\}$$

which is Jordan equivalent with the pair

$$\{\mathrm{diag}(1-a(c-b)^{-1},1,1),\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\} \tag{4.1}$$

Hence $\{A,B\}$ is in the $\mathrm{Aut}\,\mathfrak{g} \times \mathrm{Aut}\,\mathfrak{a}$ - orbit $\Omega(\tau)$ of the pair in (4.1) with $\tau = 1-a(c-b)^{-1}$.
This gives the one-parameterfamily of Lie algebras $\mathfrak{g}_{5,24}(\tau) = \mathfrak{a} \times_\theta \mathfrak{g}$.

4.1.2. Assume $b = c$. Then

$$\{A,B\} \sim \{A,a^{-1}(B-bA)\} = \{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ \mathrm{diag}(1,0,0)\},$$

and the latter pair is seen to generate the direct product $\mathfrak{g}(\theta) = \mathfrak{g}_2 \times \mathfrak{g}_{3,3}$.

4.2. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. The commutant of A consists of all matrices $B = \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}$. Here $B - aA$ is nilpotent. Hence this case is omitted (Theorem 1, (1.2)).

<u>4.3.1.</u> Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{pmatrix}$, $\alpha \neq 1$. The commuting matrices are of the form $B = \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & 0 & b \end{pmatrix}$, and we assume $\alpha a \neq b$, otherwise $a^{-1}B - A$ is nilpotent.

<u>4.3.1.1.</u> Let $\alpha = 0$. Then $b \neq 0$ and

$$\{A,B\} \sim \{A, b^{-1}B - b^{-1}cA\} = \{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\}$$

where $\beta = -cb^{-1}$. Put $\theta(e_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\theta(e_2) = \begin{pmatrix} \beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

This gives the one-parameterfamily of Lie algebras $\mathcal{G}_{5,25}(\beta) = \mathfrak{A} \times_\theta \mathcal{G}$ of 4.1.1.

<u>4.3.1.2.</u> Let $\alpha \neq 0$, and $b \neq c$. Then

$$\{A,B\} \sim \{\frac{b}{c-b} A + \frac{1}{c-b} B, \frac{1}{c-b}(cA-B)\} = \{\begin{pmatrix} \beta & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \gamma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\}$$

where $\beta = \frac{a-\alpha b}{c-b}$ and $\gamma = \frac{\alpha c - a}{c-b}$.

Hence

$$\{A,B\} \sim \{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \beta^{-1} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \gamma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\} \overset{J}{\sim} \{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \gamma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\}$$

Thus if $\theta(e_1) = A$, $\theta(e_2) = B$ then θ is in one of the Aut $\mathcal{G} \times$ Aut $\mathfrak{A}$-orbits corresponding to 4.1.1.

<u>4.3.1.3.</u> Let $\alpha \neq 0$, and $b = c$,

If $b = c = 0$ then $\{aA-B, a^{-1}B\}$ gives rise to a Lie algebra which is a direct product of two non-trivial Lie algebras. If $b \neq 0$ and $\alpha = b/a$ then $aA-B = 0$, hence is nilpotent, and this case can be omitted. If $b \neq 0$ and $\alpha \neq b/a$ then

$$\{A,B\} \sim \{\frac{b}{b-\alpha a}(A - \alpha b^{-1}B), (b-\alpha a)^{-1}(B-\alpha A)\}$$

and if $\alpha = 1$ the last pair is of the form

$$\{\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&1&1\\0&0&1\end{pmatrix}\},$$

which is seen to determine a direct product of two Lie algebras. If $\alpha \neq 1$ we obtain

$$\{A,B\} \sim \{\begin{pmatrix}t&0&0\\0&0&1\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&1&1\\0&0&1\end{pmatrix}\} \sim \{\begin{pmatrix}1&0&0\\0&0&1\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&1&1\\0&0&1\end{pmatrix}\}$$

and this leads to the case 4.1.1 , or $\mathcal{G}_{5,24}(\tau)$.

Next if $b = c \neq 0$, and $a = 0$, then

$$\{A,B\} \sim \{A-\alpha b^{-1}B, \frac{b^{-1}}{1-\alpha}B-\frac{1}{1-\alpha}A\} = \{\begin{pmatrix}1&0&0\\0&0&(1-\alpha)^{-1}\\0&0&0\end{pmatrix},\begin{pmatrix}-(1-\alpha)^{-1}&0&0\\0&1&0\\0&0&1\end{pmatrix}\}.$$

Thus we are in the $GL(2,R) \times GL(3,R)$ orbits of 4.1.1. This completes the discussion in case $\alpha \neq 1$.

<u>4.3.2</u>. Let $A = \begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}$. The matrices commuting with A are of the form $B = \begin{pmatrix}a&0&c\\d&e&f\\0&0&e\end{pmatrix}$, where $d \neq 0$ or $a \neq e$.

<u>4.3.2.1</u>. Assume $e \neq f$. Then

$$\{A,B\} \sim \{A,(e-f)^{-1}(B-fA)\} .$$

which is of the form

$$\{\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix},\begin{pmatrix}\beta&0&\varepsilon\\\delta&1&0\\0&0&1\end{pmatrix}\},\quad \beta = \frac{a-f}{e-f},\ \varepsilon = \frac{c}{e-f},\ \delta = \frac{d}{e-f} .$$

Now let $\psi = \begin{pmatrix}1&0&z\\u&1&0\\0&0&1\end{pmatrix}$. Then $\psi^{-1} = \begin{pmatrix}1&0&-z\\-u&1&uz\\0&0&1\end{pmatrix}$, and

$$\psi A\psi^{-1} = A ,\quad \psi\begin{pmatrix}\beta&0&\varepsilon\\\delta&1&0\\0&0&1\end{pmatrix}\psi^{-1} = \begin{pmatrix}\beta&0&z(1-\beta)+\varepsilon\\(\beta-1)u+\delta&1&(1-\beta)uz+\varepsilon u-\delta z\\0&0&1\end{pmatrix}$$

Since $a \neq e$ we have $\beta \neq 1$. Put

$$(\beta-1)u+\delta = 0$$

$$(1-\beta)z+\varepsilon = 0$$

Then $u = \frac{\delta}{1-\beta}$, $z = \frac{\varepsilon}{\beta-1}$, and

$$\psi\begin{pmatrix}\beta & 0 & \varepsilon\\ \delta & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\psi^{-1} = \begin{pmatrix}\beta & 0 & 0\\ 0 & 1 & \varepsilon u\\ 0 & 0 & 1\end{pmatrix} = C .$$

This means that A and B can be brought simultaneously on Jordan normal form via $\operatorname{Aut}\mathfrak{g} \times \operatorname{Aut}\mathfrak{A}$.

Now, if $\varepsilon u \neq 1$ then

$$\{A,C\} \sim \{(\beta-1)^{-1}(C-A),(1-\rho)^{-1}(C-\rho A) \quad , \text{ where } \quad \rho = \varepsilon u ,$$

and the latter pair is of the form

$$\{\begin{pmatrix}\tau & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & \sigma\\ 0 & 0 & 0\end{pmatrix}\} ,$$

which can be transformed into

$$\underset{\sim}{J} \{\begin{pmatrix}\tau & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}\} ,$$

conjugating with $\psi = \begin{pmatrix}1 & 0 & 0\\ 0 & \sigma^{-1} & 0\\ 0 & 0 & 0\end{pmatrix}$. Thus we are reduced to the case 4.1.1.

Next, if $\varepsilon u = 1$ then

$$\{A,B\} \sim \{A-(\beta-1)^{-1}(C-A),C-A\} = \{\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix},\begin{pmatrix}\beta-1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\}$$

which is seen to generate the reducible Lie algebra

$$\mathfrak{g}(\theta) = \mathfrak{g}_2 \times \mathfrak{g}_{3,3} .$$

4.3.2.2. Assume $e = f$. Then since $a \neq e$,

$$\{A,B\} \sim \{A-(a-e)^{-1}(B-eA),(a-e)^{-1}(B-eA)\} = \{\begin{pmatrix}0&0&0\\0&1&1\\0&0&1\end{pmatrix} \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}\}.$$

This again leads to the reducible Lie algebra $\mathcal{G}_2 \times \mathcal{G}_{3,3}$.

4.4. Let $A = \begin{pmatrix}1&0&0\\0&\alpha&0\\0&0&\beta\end{pmatrix} = \mathrm{diag}(1,\alpha,\beta)$.

For convenience we split the discussion into several cases.

4.4.1. Assume $\alpha = \beta \neq 1$. If $\alpha = 0$ we have $A = \mathrm{diag}(1,0,0)$, and the commutant of A consists of all matrices $B = \begin{pmatrix}a&0&0\\0&b&c\\0&d&e\end{pmatrix}$. Now if $a = 0$, the representation θ given by $\theta(e_1) = A$, $\theta(e_2) = B$ generates a Lie algebra $\mathcal{G}(\theta) = (\mathcal{G}_1)^3 \times_\theta (\mathcal{G}_1)^2$ which splits into a (nontrivial) direct product. Hence we assume $a \neq 0$. It suffices to consider Jordan normal forms A.
Thus if $B = \begin{pmatrix}1&0&0\\0&0&1\\0&0&0\end{pmatrix}$ then

$$\{A,B\} \sim \{\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}\}$$

where the second matrice is nilpotent. Hence this case is omitted, (2.1). Similarly, if

$$B = \begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix} \text{ or } \begin{pmatrix}1&0&0\\0&\beta&\gamma\\0&-\gamma&\beta\end{pmatrix}$$

then $\{A,B\}$ gives rise to Lie algebras that are direct products.

Next, assume $\alpha \neq 0$ and $\alpha \neq 1$. Then we may write $A = \mathrm{diag}(\alpha,1,1)$, via multiplication with a scalar. Now the commutant of A consists of all $B = \begin{pmatrix}a&0&0\\0&b&c\\0&d&e\end{pmatrix}$. Since A is fixed under conjugation with matrices $\psi = \begin{pmatrix}1&0&0\\0&x&y\\0&u&v\end{pmatrix}$, we need only consider

Jordan normal forms B. Omitting nilpotent matrices we have the following four cases for B,

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & 1 \\ 0 & -1 & \beta \end{pmatrix}. \tag{4.2}$$

Now the three first cases of (4.2) are contained in 4.1 and 4.3, and they all give the same $GL(2,R) \times GL(3,R)$ orbit $\Omega(\alpha)$, containing the pair

$$\left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \qquad \alpha \neq 0 .$$

Next if $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & 1 \\ 0 & -1 & \beta \end{pmatrix}$ we obtain orbits $\Omega(\alpha,\beta)$ with $\Omega(\alpha,\beta) \neq \Omega(\alpha',\beta')$ iff $(\alpha,\beta) \neq (\alpha',\beta')$.

<u>4.4.2</u>. Let $A = \mathrm{diag}(1,1,\beta)$. If $\beta \neq 0$ this is contained in 4.4.1. If $\beta = 0$ then the commuting matrices are $B = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{pmatrix}$. We assume $e \neq 0$, otherwise the corresponding Lie algebras are direct products of proper subalgebras. Write B as a Jordan normal form. Then one has as in 4.1 and 4.3,

$$\{A,B\} \sim \{A, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

whenever all the eigenvalues of B are real. When $B = \begin{pmatrix} \beta & 1 & 0 \\ -1 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ we obtain the orbit $\Omega(0,\beta)$ of 4.4.1.

<u>4.4.3</u>. Let $A = \mathrm{diag}(1,\alpha,\beta)$, where $\alpha \neq \beta$ and $\alpha \neq 1,0; \beta \neq 1,0$. The commuting matrices are $B = \mathrm{diag}(a,b,c)$. Then, if $\beta a \neq c$,

$$\{A,B\} \sim \{A,(aA-B)(\beta a-c)^{-1}\} = \{A,\mathrm{diag}(0,t,1)\}$$

Note that $\beta a \neq c$ or $\alpha a \neq b$, otherwise $aA-B = 0$.

Hence we may assume $\beta a - c \neq 0$. Further

$$\{A,B\} \sim \{\mathrm{diag}(1,\alpha-\beta t,0),\ \mathrm{diag}(0,t,1)\}$$
$$\sim \{\mathrm{diag}(s,1,0),\ \mathrm{diag}(0,1,t)\}$$

where $t = \frac{\alpha a - c}{\beta a - c}$ and $s = (\alpha - \beta t)^{-1} = \frac{a\beta - c}{(\alpha-\beta)c}$.

<u>4.4.4</u>. Finally, if $A = I$ then any B commutes, and we may obviously assume B is of Jordan normal form. Hence this case is contained in 4.1 - 4.4 above, and in 4.5 below.

<u>4.5</u>. Assume $A = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 1 \\ 0 & -1 & \beta \end{pmatrix}$. The commuting matrices are of the form $B = \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & -c & b \end{pmatrix}$.

<u>4.5.1</u>. Let $a = 0$. If $c = 0$ and $b \neq 0$ we can write, via multiplication with b^{-1}, $B = \mathrm{diag}(0,1,1)$. Hence

$$\{A,B\} \sim \{\begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\}.$$

Next, if $b = 0$ and $c \neq 0$, we may assume

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Hence, if $\beta \neq 0$,

$$\{A,B\} \sim \{\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\} \sim \{\mathrm{diag}(\alpha,1,1), \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\}$$

If $\beta = 0$, then $\{A,B\}$ generates a direct product of two nontrivial Lie algebras.

Finally if $b \neq 0$ and $c \neq 0$, we obtain

$$\{A,B\} \sim \{\begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & d & 1 \\ 0 & -1 & d \end{pmatrix}\},$$

whenever $\beta \neq 0$. Again $\beta = 0$ gives a direct product. The three cases above can be subsumed in the case $\{\mathrm{diag}(\alpha,1,1), \begin{pmatrix} \beta & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\}$.

4.5.2. Let $a \neq 0$. If $\alpha = 0$ we are reduced to the case 4.5.1. On the other hand, if $\alpha \neq 0$, we can reduce the case to 4.5.1 by considering the pair $\{A, \alpha^{-1} a A - B\}$.

4.6. We summarize the results of 4.1 - 4.5 as follows. A representation θ of the Lie algebra $\mathcal{G}$ is said to be a representation of non-nilpotent operators provided $\theta(x)$ is nilpotent iff $x = 0$, $(x \in \mathcal{G})$.

4.6.1. Proposition. Let $\theta : (\mathcal{G}_1)^2 \to \mathrm{End}(\mathcal{G}_1)^3$ be a representation of non-nilpotent operators. Assume the semidirect product of Lie algebras $(\mathcal{G}_1)^3 \times_\theta (\mathcal{G}_1)^2$ contains no nontrivial direct factors. Let $(e_i)_{i=1}^2 = \mathcal{E}$ and $(e_i)_{i=3}^5 = \mathcal{F}$ be basis for $(\mathcal{G}_1)^2$ and $(\mathcal{G}_1)^3$ respectively.

(1) Suppose $\theta(x)$ has only real eigenvalues for all x in $(\mathcal{G}_1)^2$. Then θ lies in the $GL(2,\mathbb{R}) \times GL(3,\mathbb{R})$-orbit of exactly one of the following families of representations.

(a) θ_α, where $\theta_\alpha(e_1) = \mathrm{diag}(\alpha,1,1)$ and $\theta_\alpha(e_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\alpha \neq 1$.

(b) $\theta_{\alpha,\beta}$, where $\theta_{\alpha,\beta}(e_1) = \mathrm{diag}(\alpha,1,0)$ and $\theta_{\alpha,\beta}(e_2) = \mathrm{diag}(0,1,\beta)$, $0 < |\alpha| \leq 1$, $1 \leq |\beta|$, $(\alpha,\beta) \neq (-1,1)$.

(2) Suppose $\theta(e_2)$ has a non-real eigenvalue. Then θ lies in the $GL(2,\mathbb{R}) \times GL(3,\mathbb{R})$-orbit of exactly one of the representations $\tau_{\alpha,\beta}$, where $\tau_{\alpha,\beta}(e_1) = \mathrm{diag}(\alpha,1,1)$ and $\tau_{\alpha,\beta}(e_2) = \begin{pmatrix} \beta & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$, $\alpha \neq 0$ or $\beta \neq 0$.

In (1) and (2) the representations are given relative to $\mathcal{E}$ and $\mathcal{F}$.

4.6.2. The representations $\theta_{\alpha,\beta}$, θ_α, and $\tau_{\alpha,\beta}$ in 4.6.1 determine within isomorphicms all the irreducible semidirect products of the abelian Lie algebras $(\mathcal{G}_1)^2$ and $(\mathcal{G}_1)^3$. They are denoted by $\mathcal{G}_{5,24}(\alpha,\beta)$, $\mathcal{G}_{5,25}(\alpha)$, and $\mathcal{G}_{5,26}(\alpha,\beta)$, respectively.

4.7. Proposition 4.6.1 classifies all irreducible semidirect products $(\mathcal{G}_1)^3 \times_\theta (\mathcal{G}_1)^2$. In order to complete our classification of solvable extensions of $(\mathcal{G}_1)^2$ by $(\mathcal{G}_1)^3$ we must determine the second cohomology groups $H^2((\mathcal{G}_1)^2,\theta)$. We start with the representations θ_α, $\theta_{\alpha,\beta}$, and $\tau_{\alpha,\beta}$ of Proposition 4.6.1.

4.7.1. Let $\theta = \theta_\alpha$, $\alpha \neq 1$. Hence we can find basis $\mathcal{E} = (e_i)_{i=1}^2$ for $(\mathcal{G}_1)^2$ and $\mathcal{F}$ for $(\mathcal{G}_1)^3$ such that $\theta(e_1) = \operatorname{diag}(\alpha,1,1)$ and $\theta(e_2) = \begin{pmatrix}1&0&0\\0&0&1\\0&0&0\end{pmatrix}$ relative to $\mathcal{F}$. We claim that $H^2((\mathcal{G}_1)^2,\theta) = (0)$. Let $f:(\mathcal{G}_1)^2 \to (\mathcal{G}_1)^3$ be linear. Writing $f = \begin{pmatrix}\varphi_1&\psi_1\\\varphi_2&\psi_2\\\varphi_3&\psi_3\end{pmatrix}$ relative to $\mathcal{E}$ and $\mathcal{F}$ we have

$$\begin{aligned} df(X,Y) &= \theta(X)f(Y) - \theta(Y)f(X) \\ &= \begin{pmatrix}\alpha\psi_1 - \varphi_1\\ \psi_2 - \varphi_3\\ \psi_3\end{pmatrix}(x_1y_2 - x_2y_1) = \begin{pmatrix}\alpha\psi_1 - \varphi_1\\ \psi_2 - \varphi_3\\ \psi_3\end{pmatrix}B_{12}(X,Y), \end{aligned}$$

where $X,Y \in (\mathcal{G}_1)^2$. Hence the space of coboundaries consists of $(B_{12}e_3, B_{12}e_4, B_{12}e_5)$ and $H^2(\mathcal{G},\theta) = (0)$.

4.7.2. Let $\theta = \theta_{\alpha,\beta}$. Hence we may assume $\theta(e_1) = \operatorname{diag}(\alpha,1,0)$, $\theta(e_2) = \operatorname{diag}(0,1,\beta)$. With notation as in 4.7.1 we have

$$df(X,Y) = \begin{pmatrix}\alpha\psi_1\\ \psi_2 - \varphi_2\\ -\beta\varphi_3\end{pmatrix}B_{12}(X,Y), \quad X,Y \in (\mathcal{G}_1)^2.$$

Hence $H^2(\mathcal{G},\theta) = (0)$ if $\beta \neq 0$ and $H^2(\mathcal{G},\theta) = (\begin{pmatrix} 0 \\ 0 \\ B_{12} \end{pmatrix})$ if $\beta = 0$. Assume therefore $\beta = 0$. Via a change of basis in $\mathcal{G}$ we have $\theta(e_1) = \mathrm{diag}(1,0,0)$, $\theta(e_2) = \mathrm{diag}(0,1,0)$, and we obtain the Lie algebra $\mathcal{G}_{5,27} = \mathcal{G}(B_{12},\theta)$. (We note that this particular θ was not allowed in 4.6.1 since it generates a reducible semidirect psoduct $(\mathcal{G}_1)^3 \times_\theta (\mathcal{G}_1)^2 \approx \mathcal{G}_2 \times \mathcal{G}_2 \times \mathcal{G}_1$.)

4.7.3. Let $\theta = \tau_{\alpha,\beta}$. Hence we assume $\theta(e_1) = \mathrm{diag}(\alpha,1,1)$, $\theta(e_2) = \begin{pmatrix} \beta & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$. Using the notation of 4.7.1 we have

$$\begin{aligned} df(X,Y) &= x_1\,\mathrm{diag}(\alpha,1,1)\begin{pmatrix} \varphi_1 & \psi_1 \\ \varphi_2 & \psi_2 \\ \varphi_3 & \psi_3 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - y_1\,\mathrm{diag}(\alpha,1,1)\begin{pmatrix} \varphi_1 & \psi_1 \\ \varphi_2 & \psi_2 \\ \varphi_3 & \psi_3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &+ x_2\begin{pmatrix} \beta & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} \varphi_1 & \psi_1 \\ \varphi_2 & \psi_2 \\ \varphi_3 & \psi_3 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - y_2\begin{pmatrix} \beta & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} \varphi_1 & \psi_1 \\ \varphi_2 & \psi_2 \\ \varphi_3 & \psi_3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} \alpha\psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} B_{12}(X,Y) - \begin{pmatrix} \beta\varphi_1 \\ \varphi_3 \\ -\varphi_2 \end{pmatrix} B_{12}(X,Y) \end{aligned}$$

Therefore $H^2(\mathcal{G},\theta) = (0)$ unless $\alpha = \beta = 0$, and $H^2(\mathcal{G},\tau_{0,0}) = (\begin{pmatrix} B_{12} \\ 0 \\ 0 \end{pmatrix})$. We obtain the Lie algebra $_{5,28} = (B_{12},\tau_{0,0})$ with bracket operations

$$[e_1,e_2] = e_3\ ,\quad [e_1,e_4] = e_4\ ,\quad [e_1,e_5] = e_5\ ,$$

$$[e_2,e_4] = e_5\ ,\quad [e_2,e_5] = -e_4\ .$$

(Note that $\tau_{0,0}$ generates a reducible semidirect product.)

4.7.4. Finally we must consider all representations $\theta:(\mathcal{G}_1)^2 \quad \mathrm{End}((\mathcal{G}_1)^3)$ which are not in the orbits of the representations θ_α, $\theta_{\alpha,\beta}$, and $\tau_{\alpha,\beta}$. Checking the sections 4.1 - 4.5 we see that the only possibility is the representation θ with $\theta(e_1) = \mathrm{diag}(1,0,0)$ and $\theta(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Computing as above one finds $H^2(\quad ,\theta) = (0)$. Hence there are no extensions.

4.7.5. Summing up, there are two non-semidirect extensions of $(\mathcal{G}_1)^2$ by $(\mathcal{G}_1)^3$ satisfying the hypothesis of Theorem 1. They are

$$\mathcal{G}_{5,27} = \mathcal{G}(B_{12},\theta),\ \theta(e_1) = \mathrm{diag}(1,0,0),\ \theta(e_2) = \mathrm{diag}(0,1,0)$$

$$\mathcal{G}_{5,28} = \mathcal{G}(B_{12},\tau_{0,0}),\ \tau_{0,0}(e_1) = \mathrm{diag}(0,1,1),\ \tau_{0,0}(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

4.8. Let $\mathcal{G} = \mathcal{G}_2$ with defining basis relations $[e_1,e_2] = e_2$. Hence the commutator and nilradical both are equal to (e_2), and we put $\mathcal{N} = (e_2)$. Let B be an alternating bilinear map on $\mathcal{G}$ with values in $(\mathcal{G}_1)^3$. It is immediate that $\mathcal{S}_{B^\circ} \cap \mathfrak{Z}(\mathcal{N}) \neq (0)$. Hence there are no $\mathcal{N}$-admissible extensions of $\mathcal{G}_2$ by $(\mathcal{G}_1)^3$.

5. Extensions of $\mathfrak{g}_1$. Let $\mathfrak{g} = \mathfrak{g}_1$, $\mathfrak{a} = (\mathfrak{g}_1)^4$. In this case $H^2(\mathfrak{g}, \mathfrak{a}) = (0)$, and all extensions of $\mathfrak{g}$ by $\mathfrak{a}$ are given as semidirect products $\mathfrak{g}(\theta) = (\mathfrak{g}_1)^4 \times_\theta \mathfrak{g}_1$, where $\theta : \mathfrak{g} \to \operatorname{End} \mathfrak{a}$ is a representation. Let $e_1 \in \mathfrak{g}_1$, $e_1 \neq 0$. Then θ is uniquely determined by its value at e_1. Hence it is sufficient to consider all possible Jordan normal forms $\theta(e_1)$. Further $\operatorname{Aut} \mathfrak{g}$ acts via multiplication by nonzero real numbers. Thus we can "normalize" the Jordan forms. Finally we omit the nilpotent matrices $\theta(e_1)$. Fix a basis $\mathcal{E} = (e_i)_{i=2}^5$ for $\mathfrak{a}$. We realize $\theta(e_1)$ relative to $\mathcal{E}$ as follows.

5.1. Assume $\theta(e_1)$ has only real eigenvalues. There are five cases to consider.

$$\begin{pmatrix} \alpha & 1 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ |\alpha| \leq 1 \ ; \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} ; \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 1 & 0 \\ 0 & 0 & \alpha & 1 \\ 0 & 0 & 0 & \alpha \end{pmatrix} ;$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 1 \\ 0 & 0 & 0 & \beta \end{pmatrix}, \quad \begin{array}{l} -1 < \alpha \leq \beta < 0 \text{, or } 0 < \alpha \leq \beta \leq 1 \text{, or } -1 \leq \alpha < 0 \ \& \\ 0 < \beta \leq 1 \text{, or } \beta = 0 \ \& \ 0 < |\alpha| \leq 1 \ ; \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix}, \quad \begin{array}{l} 0 < \alpha \leq \beta \leq \gamma \leq 1 \text{, or } -1 < \alpha \leq \beta \leq \gamma < 0 \text{, or} \\ -1 \leq \alpha \leq \beta < 0 \ \& \ 0 < \gamma \leq 1 \text{, or } -1 \leq \alpha < 0 \ \& \ 0 < \beta \leq \gamma \leq 1 . \end{array}$$

This gives the five families of Lie algebras

$$\mathfrak{g}_{5,29}(\alpha), \ \mathfrak{g}_{5,30}, \ \mathfrak{g}_{5,31}(\alpha), \ \mathfrak{g}_{5,32}(\alpha,\beta), \text{ and } \mathfrak{g}_{5,33}(\alpha,\beta,\gamma)$$

respectively.

5.2. $\theta(e_1)$ has complex eigenvalues. There are three cases to consider.

$$\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 1 \\ 0 & 0 & -1 & \gamma \end{pmatrix}, \quad \alpha > 0 \ , \quad 0 < |\beta| \leq 1$$

$$\begin{pmatrix} \alpha & 1 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 1 \\ 0 & 0 & -1 & \beta \end{pmatrix}, \quad \alpha \geq 0$$

$$\begin{pmatrix} \alpha & 1 & 0 & 0 \\ -1 & \alpha & 0 & 0 \\ 0 & 0 & \beta & \gamma \\ 0 & 0 & -\gamma & \beta \end{pmatrix}, \quad \alpha \geq 0 \ , \quad \gamma > 0 \ .$$

From the above we obtain the three following families of pairwise nonisomorphic Lie algebras,

$$\mathcal{G}_{5,34}(\alpha,\beta,\gamma), \quad \mathcal{G}_{5,35}(\alpha,\beta), \text{ and } \quad \mathcal{G}_{5,36}(\alpha,\beta,\gamma)$$

respectively.

This completes our construction of real sovable Lie algebras of dimension five.

6.1. The case $\dim \mathfrak{g} = 4$, $\dim \mathfrak{a} = 1$, $F = \mathbb{R}$

$\mathfrak{g}$	$\mathfrak{n}$	Representation θ Cocycle B	Lie products in $\mathfrak{g}$	Lie products in extension $\mathfrak{g}(B,\theta)$	$\mathfrak{g}(B,\theta)$
$(\mathfrak{g}_1)^4$	$(\mathfrak{g}_1)^4$	$\theta = 0$ $B_{12} + B_{23}$	0	$[e_1,e_2] = e_5$ $[e_3,e_4] = e_5$	$\mathfrak{n}_{5,1}$
$\mathfrak{g}_2 \times \mathfrak{g}_2$	$(\mathfrak{g}_1)^2$ (e_2,e_4)	$\theta(e_1)e_5 = e_5$ $\theta(e_3)e_5 = e_5$ B_{24}	$[e_1,e_2] = e_2$ $[e_3,e_4] = e_4$	$[e_1,e_5] = e_5 = [e_3,e_5]$ $[e_2,e_4] = e_5$	$\mathfrak{g}_{5,1}$
$\mathfrak{g}_{3,1} \times \mathfrak{g}_1$	$\mathfrak{g}_{3,1} \times \mathfrak{g}_1$	$\theta = 0$ $B_{14} + B_{23}$	$[e_1,e_2] = e_3$	$[e_1,e_4] = e_5$ $[e_2,e_3] = e_5$	$\mathfrak{n}_{5,3}$
$\mathfrak{g}_{4,1}$	$\mathfrak{g}_{3,1}$ (e_2,e_3,e_4)	$\theta(e_1)e_5 = 2e_5$ B_{34}		$[e_1,e_5] = 2e_5$ $[e_3,e_4] = e_5$	$\mathfrak{g}_{5,2}$
		$\theta(e_1)e_5 = e_5$ B_{24}	$[e_1,e_3] = e_3$ $[e_1,e_4] = e_4$ $[e_2,e_3] = e_4$	$[e_1,e_5] = e_5$ $[e_2,e_4] = e_5$	$\mathfrak{g}_{5,3}$
		$\theta(e_1)e_5 = e_5$ $B_{24} + B_{13}$	$[e_2,e_3] = e_4$ $[e_1,e_4] = e_4$	$[e_1,e_5] = e_5$ $[e_1,e_3] = e_3 + e_5$ $[e_2,e_4] = e_5$	$\mathfrak{g}_{5,4}$
		$\theta(e_1)e_5 = e_5$ $B_{24} - B_{13}$	$[e_2,e_3] = e_4$ $[e_1,e_4] = e_4$	$[e_1,e_5] = e_5$ $[e_1,e_3] = e_3 - e_5$ $[e_2,e_4] = e_5$	$\mathfrak{g}_{5,5}$
$\mathfrak{g}_{4,2}$	$(\mathfrak{g}_1)^2$ (e_3,e_4)	$\theta(e_1)e_5 = 2e_5$ $\theta(e_2) = 0$ B_{34}	$[e_1,e_3] = e_3$ $[e_1,e_4] = e_4$ $[e_2,e_3] = -e_4$ $[e_2,e_4] = e_3$	$[e_1,e_5] = 2e_5$ $[e_3,e_4] = e_5$	$\mathfrak{g}_{5,6}$
$\mathfrak{g}_{4,3}$	$\mathfrak{g}_{4,3}$	$\theta = 0$, B_{14}	$[e_1,e_2] = e_3$ $[e_1,e_3] = e_4$	$[e_1,e_4] = e_5$	$\mathfrak{n}_{5,5}$
		$\theta = 0$, $B_{14} + B_{23}$		$[e_1,e_4] = [e_2,e_3] = e_5$	$\mathfrak{n}_{5,6}$
$\mathfrak{g}_{4,9}(\alpha)$ $0 \le \alpha \le 2$ $\alpha \ne 1$	$\mathfrak{g}_{3,1}$ (e_2,e_3,e_4)	$\alpha \ne 0,2$: $\theta(e_1)e_5 = (2\alpha-1)e_5$ B_{24}	$[e_1,e_2] = (\alpha-1)e_2$ $[e_1,e_3] = e_3$ $[e_1,e_4] = \alpha e_4$ $[e_2,e_3] = e_4$	$[e_1,e_5] = (2\alpha-1)e_5$ $[e_2,e_4] = e_5$	$\mathfrak{g}_{5,7}(\alpha)$ $0 < \alpha \le 2$ $\alpha \ne 1$
		$\theta(e_1)e_5 = (\alpha+1)e_5$ B_{34}		$[e_1,e_5] = (\alpha+1)e_5$ $[e_3,e_4] = e_5$	$\mathfrak{g}_{5,8}(\alpha)$ $0 \le \alpha \le 2$ $\alpha \ne 1$
$\mathfrak{g}_{4,10}$	$\mathfrak{g}_3$	$\theta(e_1)e_5 = 3e_5$ B_{24}	$[e_1,e_2] = e_2 + e_3$ $[e_1,e_3] = e_3$ $[e_1,e_4] = e_4$ $[e_2,e_3] = e_4$	$[e_1,e_5] = 3e_5$ $[e_2,e_4] = e_5$	$\mathfrak{g}_{5,9}$

6.2. The case $\dim \mathcal{G} = 3$, $\dim \mathfrak{a} = 2$, $F = \mathbb{R}$

$\mathcal{G}$	$\mathfrak{n}$	Repres. θ cocycle B	Lie Products in extension $\mathcal{G}(B,\theta)$		$\mathcal{G}(B,\theta)$
			Lie products in $\mathcal{G}$		
$(\mathcal{G}_1)^3$	$(\mathcal{G}_1)^3$ (e_1,e_2,e_3)	$\theta = 0$ (B_{12},B_{13})	0	$[e_1,e_2]=e_4$ $[e_1,e_3]=e_5$	$\mathfrak{n}_{5,2}$
$\mathcal{G}_2 \times \mathcal{G}_1$	$(\mathcal{G}_1)^2$ (e_2,e_3)	$\theta(e_1)=\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}$ $(0,B_{23})$	$[e_1,e_2]=e_2$	$[e_1,e_4]=\alpha e_4$ $[e_1,e_5]=e_5$ $[e_2,e_3]=e_5$	$\mathcal{G}_{5,10}(\alpha)$
		$\theta(e_1)=\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}$ (B_{13},B_{23})		$[e_1,e_4]=\alpha e_4$ $[e_1,e_5]=e_5$ $[e_1,e_3]=e_4$ $[e_2,e_3]=e_5$	$\mathcal{G}_{5,11}(\alpha)$
		$\theta(e_1)=I$ (B_{23},B_{12})	$[e_1,e_2]=e_2+e_5$	$[e_1,e_4]=e_4$ $[e_1,e_5]=e_5$ $[e_2,e_3]=e_4$	$\mathcal{G}_{5,12}$
		$\theta(e_1)=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ $(B_{23},0)$	$[e_1,e_2]=e_2$	$[e_1,e_4]=e_4$ $[e_1,e_5]=e_5+e_4$ $[e_2,e_3]=e_4$	$\mathcal{G}_{5,13}$
		$\theta(e_1)=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ (B_{23},B_{12})	$[e_1,e_2]=e_2+e_5$	$[e_1,e_4]=e_4$ $[e_1,e_5]=e_4+e_5$ $[e_2,e_3]=e_4$	$\mathcal{G}_{5,14}$

(dim $\mathcal{G}$ = 3 , dim $\mathfrak{a}$ = 2 , continued)

$\mathcal{G}$	$\mathfrak{n}$	Repres. θ cocycle B	Lie products in extension $\mathcal{G}(B,\theta)$: Lie products in $\mathcal{G}$		$\mathcal{G}(B,\theta)$
$\mathcal{G}_{3,1}$	$\mathcal{G}_{3,1}$ (e_1,e_2,e_3)	$\theta = 0$ (B_{13},B_{14})	$[e_1,e_2]=e_3$	$[e_1,e_3]=e_4$ $[e_1,e_4]=e_5$	$\mathfrak{n}_{5,4}$
	$(\mathcal{G}_1)^2$ (e_2,e_3)	$\theta(e_1)=\begin{pmatrix}0&0\\0&1\end{pmatrix}$ $(B_{23},0)$		$[e_1,e_5]=e_5$ $[e_2,e_3]=e_4$	$\mathcal{G}_{5,15}$
$\mathcal{G}_{3,2}(\alpha)$ $\lvert\alpha\rvert \geq 1$	$(\mathcal{G}_1)^2$ (e_2,e_3)	$\theta(e_1)=\begin{pmatrix}\alpha+1&0\\0&1\end{pmatrix}$ (B_{23},B_{12})	$[e_1,e_2]=e_2+e_5$ $[e_1,e_3]=\alpha e_3$	$[e_1,e_4]=(\alpha+1)e_4$ $[e_1,e_5]=e_5$ $[e_2,e_3]=e_4$	$\mathcal{G}_{5,16}(\alpha)$ $\lvert\alpha\rvert \geq 1$
		$\theta(e_1)=\begin{pmatrix}\alpha+1&0\\0&\beta\end{pmatrix}$ $(B_{23},0)$	$[e_1,e_2]=e_2$ $[e_1,e_3]=\alpha e_3$	$[e_1,e_4]=(\alpha+1)e_4$ $[e_1,e_5]=\beta e_5$ $[e_2,e_3]=e_4$	$\mathcal{G}_{5,17}(\alpha,\beta)$ $\lvert\alpha\rvert \geq 1, \beta\neq 0$
		$\theta(e_1)=\begin{pmatrix}\alpha+1&1\\0&\alpha+1\end{pmatrix}$ $(B_{23},0)$		$[e_1,e_4]=(\alpha+1)e_4$ $[e_1,e_5]=e_4+(\alpha+1)e_5$ $[e_2,e_3]=e_4$	$\mathcal{G}_{5,18}(\alpha)$ $\lvert\alpha\rvert\geq 1$
$\mathcal{G}_{3,3}$	$(\mathcal{G}_1)^2$ (e_2,e_3)	$\theta(e_1)=\begin{pmatrix}2&0\\0&\alpha\end{pmatrix}$ $(B_{23},0)$	$[e_1,e_2]=e_2+e_3$ $[e_1,e_3]=e_3$	$[e_1,e_4]=2e_4$ $[e_1,e_5]=\alpha e_5$ $[e_2,e_3]=e_4$	$\mathcal{G}_{5,19}(\alpha)$ $\alpha\neq 0$
		$\theta(e_1)=\begin{pmatrix}2&0\\0&1\end{pmatrix}$ (B_{23},B_{13})	$[e_1,e_2]=e_2+e_3$ $[e_1,e_3]=e_3+e_5$	$[e_1,e_4]=2e_4$ $[e_1,e_5]=e_5$ $[e_2,e_3]=e_4$	$\mathcal{G}_{5,20}$
		$\theta(e_1)=\begin{pmatrix}2&1\\0&2\end{pmatrix}$ $(B_{23},0)$		$[e_1,e_4]=2e_4+e_5$ $[e_1,e_5]=2e_5$ $[e_2,e_3]=e_4$	$\mathcal{G}_{5,21}$
$\mathcal{G}_{3,4}(\alpha)$ $\alpha\geq 0$	$(\mathcal{G}_1)^2$ (e_2,e_3)	$\theta(e_1)=\begin{pmatrix}2\alpha&0\\0&\beta\end{pmatrix}$ $(B_{23},0)$	$[e_1,e_2]=\alpha e_2-e_3$ $[e_1,e_3]=e_2+\alpha e_3$	$[e_1,e_4]=2\alpha e_4$ $[e_1,e_5]=\beta e_5$ $[e_2,e_3]=e_4$	$\mathcal{G}_{5,22}(\alpha,\beta)$ $\alpha>0$ or $\beta\neq 0$
		$\theta(e_1)=\begin{pmatrix}2\alpha&1\\0&2\alpha\end{pmatrix}$ $(B_{23},0)$		$[e_1,e_4]=2\alpha e_4$ $[e_1,e_5]=e_4+2\alpha e_5$ $[e_2,e_3]=e_4$	$\mathcal{G}_{5,23}(\alpha)$ $\alpha\geq 0$

6.3. The case $\dim \mathcal{G} = 2$, $\dim \mathfrak{a} = 3$

We first list all semidirect products $(\mathcal{G}_1)^3 \times_\theta (\mathcal{G}_1)^2$.

$\theta(e_1)$	$\theta(e_2)$	Lie products in semidirect product $(\mathcal{G}_1)^3 \times_\theta (\mathcal{G}_1)^2$	$\mathcal{G}(\theta)$
$\begin{pmatrix} \alpha & & \\ & 1 & \\ & & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & & \\ & 1 & \\ & & \beta \end{pmatrix}$	$[e_1,e_3]=\alpha e_3$ $[e_1,e_4]=e_4$ $[e_2,e_4]=e_4$ $[e_2,e_5]=\beta e_5$	$\mathcal{G}_{5,24}(\alpha,\beta)$ $0<\lvert\alpha\rvert\leq 1$ $1\leq\lvert\beta\rvert$, $(\alpha,\beta)\neq(-1,1$
$\begin{pmatrix} \alpha & & \\ & 1 & \\ & & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$[e_1,e_3]=\alpha e_3$ $[e_1,e_4]=e_4$ $[e_1,e_5]=e_5$ $[e_2,e_3]=e_3$ $[e_2,e_5]=e_4$	$\mathcal{G}_{5,25}(\alpha)$ $\alpha\neq 1$
	$\begin{pmatrix} \beta & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$	$[e_1,e_3]=\alpha e_3$ $[e_1,e_4]=e_4$ $[e_1,e_5]=e_5$ $[e_2,e_3]=\beta e_3$ $[e_2,e_4]=-e_5$ $[e_2,e_5]=e_4$	$\mathcal{G}_{5,26}(\alpha,\beta)$ $\alpha\neq 0$ or $\beta\neq 0$

The remaining extensions of $(\mathcal{G}_1)^2$ by $(\mathcal{G}_1)^3$ are listed below:

$\mathcal{G}$	$\mathfrak{n}$	Representation $\theta(e_1)$	$\theta(e_2)$	Cocycle B	Lie products in extension $\mathcal{G}(B,\theta)$	$\mathcal{G}(B,\theta)$
$(\mathcal{G}_1)^2$	(0)	$\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ B_{12} \end{pmatrix}$	$[e_1,e_3]=e_3$ $[e_1,e_2]=e_5$ $[e_2,e_4]=e_4$	$\mathcal{G}_{5,27}$
		$\begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} B_{12} \\ 0 \\ 0 \end{pmatrix}$	$[e_1,e_4]=e_4$ $[e_1,e_5]=e_5$ $[e_2,e_4]=-e_5$ $[e_2,e_5]=e_4$ $[e_1,e_2]=e_3$	$\mathcal{G}_{5,28}$

6.4. The case dim $\mathcal{G} = 1$, dim $\mathfrak{A} = 4$, $F = \mathbb{R}$

Jordan forms $\theta(e_1)$	Lie products in semidirect product $\mathcal{G}(\theta) = (\mathcal{G}_1)^4 \times_\theta \mathcal{G}_1$	$\mathcal{G}(\theta)$
$\begin{pmatrix}\alpha & 1 & 0 & 0\\ 0 & \alpha & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\end{pmatrix}$	$[e_1,e_2]=\alpha e_2$ $[e_1,e_3]=e_2+\alpha e_3$ $[e_1,e_4]=e_4$ $[e_1,e_5]=e_4+e_5$	$\mathcal{G}_{5,29}(\alpha)$ $\lvert\alpha\rvert \leq 1$
$\begin{pmatrix}1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\end{pmatrix}$	$[e_1,e_2]=e_2$ $[e_1,e_3]=e_2+e_3$ $[e_1,e_4]=e_3+e_4$ $[e_1,e_5]=e_4+e_5$	$\mathcal{G}_{5,30}$
$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \alpha & 1 & 0\\ 0 & 0 & \alpha & 1\\ 0 & 0 & 0 & \alpha\end{pmatrix}$	$[e_1,e_2]= e_2$ $[e_1,e_3]= \alpha e_3+e_4$ $[e_1,e_4]= \alpha e_4+e_5$ $[e_1,e_5]=\alpha e_5$	$\mathcal{G}_{5,31}(\alpha)$
$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \alpha & 0 & 0\\ 0 & 0 & \beta & 1\\ 0 & 0 & 0 & \beta\end{pmatrix}$	$[e_1,e_2]=e_2$ $[e_1,e_3]=\alpha e_3$ $[e_1,e_4]=\beta e_4$ $[e_1,e_5]=e_4+\beta e_5$	$\mathcal{G}_{5,32}(\alpha,\beta)$ $-1<\alpha\leq\beta<0$, or $0<\alpha\leq\beta\leq 1$, or $-1\leq\alpha<0$ & $0<\beta\leq 1$, or $\beta=0$ & $0<\lvert\alpha\rvert\leq 1$.
$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \alpha & 0 & 0\\ 0 & 0 & \beta & 0\\ 0 & 0 & 0 & \gamma\end{pmatrix}$	$[e_1,e_2]=e_2$ $[e_1,e_3]=\alpha e_3$ $[e_1,e_4]=\beta e_4$ $[e_1,e_5]=\gamma e_5$	$\mathcal{G}_{5,33}(\alpha,\beta,\gamma)$ $0<\alpha\leq\beta\leq\gamma\leq 1$, or $-1<\alpha\leq\beta\leq\gamma<0$, or $-1\leq\alpha\leq\beta<0$ & $0<\gamma\leq 1$, or $-1\leq\alpha<0$ & $0<\beta\leq\gamma\leq 1$.
$\begin{pmatrix}\alpha & 0 & 0 & 0\\ 0 & \beta & 0 & 0\\ 0 & 0 & \gamma & 1\\ 0 & 0 & -1 & \gamma\end{pmatrix}$	$[e_1,e_2]=\alpha e_2$ $[e_1,e_3]=\beta e_3$ $[e_1,e_4]=\gamma e_4-e_5$ $[e_1,e_5]=e_4+\gamma e_5$	$\mathcal{G}_{5,34}(\alpha,\beta,\gamma)$ $\alpha>0$, $0<\lvert\beta\rvert\leq 1$
$\begin{pmatrix}\alpha & 1 & 0 & 0\\ 0 & \alpha & 0 & 0\\ 0 & 0 & \beta & 1\\ 0 & 0 & -1 & \beta\end{pmatrix}$	$[e_1,e_2]=\alpha e_2$ $[e_1,e_3]=e_2+\alpha e_3$ $[e_1,e_4]=\beta e_4-e_5$ $[e_1,e_5]=e_4+\beta e_5$	$\mathcal{G}_{5,35}(\alpha,\beta)$ $\alpha\geq 0$
$\begin{pmatrix}\alpha & 1 & 0 & 0\\ -1 & \alpha & 0 & 0\\ 0 & 0 & \beta & \gamma\\ 0 & 0 & -\gamma & \beta\end{pmatrix}$	$[e_1,e_2]=\alpha e_2-e_3$ $[e_1,e_3]=e_2+\alpha e_3$ $[e_1,e_4]=\beta e_4-\gamma e_5$ $[e_1,e_5]=\gamma e_4+\beta e_5$	$\mathcal{G}_{5,36}(\alpha,\beta,\gamma)$ $\alpha\geq 0$, $\gamma>0$

6.5. Roots of five dimensional solvable Lie algebras over $\mathbb{R}$.

$\rho\ \rho_1\ \rho_2$ denotes linearly independent linear functionals on the Lie algebra $\tilde{\mathfrak{g}}$.

$\tilde{\mathfrak{g}}$	Roots	$\tilde{\mathfrak{g}}$	Roots
$\mathfrak{g}_{5,1}$	$o,\rho_1,\rho_1,\rho_2,\rho_2$	$\mathfrak{g}_{5,24}(\alpha,\beta)$	$o,o,\alpha\rho_1,\rho_1+\rho_2,\beta\rho_2$
$\mathfrak{g}_{5,2}$	$o,o,\rho,\rho,2\rho$	$\mathfrak{g}_{5,25}(\alpha)$	$o,o,\alpha\rho,\rho,\rho$
$\mathfrak{g}_{5,3}$	o,o,ρ,ρ,ρ	$\mathfrak{g}_{5,26}(\alpha,\beta)$	$o,o,\alpha\rho_1+\beta\rho_2,\rho_1+i\rho_2,\rho_1-i\rho_2$
$\mathfrak{g}_{5,4}$	o,o,ρ,ρ,ρ		
$\mathfrak{g}_{5,5}$	o,o,ρ,ρ,ρ	$\mathfrak{g}_{5,27}$	o,o,ρ_1,ρ_2,o
$\mathfrak{g}_{5,6}$	$o,o,\rho_1+i\rho_2,\rho_1-i\rho_2,\rho_1$	$\mathfrak{g}_{5,28}$	$o,o,o,\rho_1+i\rho_2,\rho_1-i\rho_2$
$\mathfrak{g}_{5,7}(\alpha)$	$o,(\alpha+1)\rho,\rho,\alpha\rho,(2\alpha-1)\rho$		
$\mathfrak{g}_{5,8}(\alpha)$	$o,(\alpha-1)\rho,\rho,\alpha\rho,(\alpha+1)\rho$	$\mathfrak{g}_{5,29}(\alpha)$	$o,\alpha\rho,\alpha\rho,\rho,\rho$
$\mathfrak{g}_{5,9}$	$o,\rho,\rho,\rho,3\rho$	$\mathfrak{g}_{5,30}$	o,ρ,ρ,ρ,ρ
$\mathfrak{g}_{5,10}(\alpha)$	$o,\rho,o,\alpha\rho,\rho$	$\mathfrak{g}_{5,31}(\alpha)$	$o,\rho,\alpha\rho,\alpha\rho,\alpha\rho$
$\mathfrak{g}_{5,11}(\alpha)$	$o,\rho,o,\alpha\rho,\rho$	$\mathfrak{g}_{5,32}(\alpha,\beta)$	$o,\rho,\alpha\rho,\beta\rho,\beta\rho$
$\mathfrak{g}_{5,12}$	o,ρ,o,ρ,ρ	$\mathfrak{g}_{5,33}(\alpha,\beta,\gamma)$	$o,\rho,\alpha\rho,\beta\rho,\gamma\rho$
$\mathfrak{g}_{5,13}$	o,ρ,o,ρ,ρ	$\mathfrak{g}_{5,34}(\alpha,\beta,\gamma)$	$o,\alpha\rho,\beta\rho,(\gamma+i)\rho,(\gamma-i)\rho$
$\mathfrak{g}_{5,14}$	o,ρ,o,ρ,ρ	$\mathfrak{g}_{5,35}(\alpha,\beta)$	$o,\alpha\rho,\alpha\rho,(\beta+i)\rho,(\beta-i)\rho$
$\mathfrak{g}_{5,15}$	o,ρ,o,ρ,ρ	$\mathfrak{g}_{5,36}(\alpha,\beta,\gamma)$	$o,(\alpha+i)\rho,(\alpha-i)\rho,(\beta+\gamma i)\rho,(\beta-\gamma i)\rho$
$\mathfrak{g}_{5,16}(\alpha)$	$o,\rho,\alpha\rho,(\alpha+1)\rho,\rho$		
$\mathfrak{g}_{5,17}(\alpha,\beta)$	$o,\rho,\alpha\rho,(\alpha+1)\rho,\beta\rho$		
$\mathfrak{g}_{5,18}(\alpha)$	$o,\rho,\alpha\rho,(\alpha+1)\rho,(\alpha+1)\rho$		
$\mathfrak{g}_{5,19}(\alpha)$	$o,\rho,\rho,2\rho,\alpha\rho$		
$\mathfrak{g}_{5,20}$	$o,\rho,\rho,2\rho,\rho$		
$\mathfrak{g}_{5,21}(\alpha)$	$o,\rho,\rho,2\rho,2\rho$		
$\mathfrak{g}_{5,22}(\alpha,\beta)$	$o,(\alpha+i)\rho,(\alpha-i)\rho,\ \alpha\rho,\beta\rho$		
$\mathfrak{g}_{5,23}(\alpha)$	$o,(\alpha+i)\rho\ ,(\alpha-i)\rho\ ,2\alpha\rho_1,2\alpha\rho_1$		

6.6. Remarks. By [1; Theorem 2.1, p. 2] a solvable Lie algebra is non-exponential if and only if some of ite roots are purely imaginary. It is then immediate from Section 5 that the only non-exponential irreducible solvable Lie algebras of dimension five are

$$\mathcal{G}_{5,22}(0,\beta),\quad \mathcal{G}_{5,23}(0),\quad \mathcal{G}_{5,34}(\alpha,\beta,0),\quad \mathcal{G}_{5,35}(\alpha,0),$$
$$\mathcal{G}_{5,36}(\alpha,0,\gamma),\text{ and }\ \mathcal{G}_{5,35}(0,\beta,\gamma),\ \gamma \neq 0 .$$

We remark that $\mathcal{G}_{5,36}(0,0,\gamma)$ gives the Mautner algebras. The only solvable Lie groups of dimension ≤ 5 that are not of type I (in the sense of von Neumann), are the Mautner groups. Finally we note that $\mathcal{G}_{5,22}(0,\beta)$ and $\mathcal{G}_{5,23}(0)$ are extensions of the non--exponential $\mathcal{G}_{3,4}(0)$, whereas the families $\mathcal{G}_{5,34}(\alpha,\beta,0)$, $\mathcal{G}_{5,35}(\alpha,0)$, $\mathcal{G}_{5,36}(\alpha,0,\gamma)$, and $\mathcal{G}_{5,36}(0,\beta,\gamma)$ are "new" in that all of them are proper extensions of exponential Lie algebras.

6.7. Automorphism groups of solvable Lie algebras $\mathcal{G}$, $\dim \mathcal{G} \leq 4$.

$\mathcal{G}$ Basis relations	Aut ($\mathcal{G}$)
$\mathcal{G}_2$ $[e_1,e_2] = e_2$	$\begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}$, $b \neq 0$
$\mathcal{G}_{3,1}$ $[e_1,e_2] = e_3$	$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ u & v & w \end{pmatrix}$, $w = ad - bc \neq 0$
$\mathcal{G}_{3,2}(\alpha)$ $[e_1,e_2] = e_2$ $[e_1,e_3] = \alpha e_3$	$\begin{pmatrix} 1 & 0 & 0 \\ u & a & 0 \\ v & 0 & d \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ u & 0 & b \\ v & c & 0 \end{pmatrix}$ $ad \neq 0$ $bc \neq 0$
$\mathcal{G}_{3,3}$ $[e_1,e_2] = e_2 + e_3$ $[e_1,e_3] = e_3$	$\begin{pmatrix} 1 & 0 & 0 \\ u & a & 0 \\ v & b & a \end{pmatrix}$, $a \neq 0$
$\mathcal{G}_{3,4}(\alpha)$ $[e_1,e_2] = \alpha e_2 - e_3$ $[e_1,e_3] = e_2 + \alpha e_3$	$\begin{pmatrix} 1 & 0 & 0 \\ u & a & b \\ v & -b & a \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ u & a & b \\ v & b & -a \end{pmatrix}$, $a^2 + b^2 \neq 0$
$\mathcal{G}_{4,1}$ $[e_1,e_2] = e_3$ $[e_2,e_3] = e_4$ $[e_1,e_4] = e_4$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ u & 0 & d & 0 \\ v & au & w & ad \end{pmatrix}$, $ad \neq 0$
$\mathcal{G}_{4,2}$ $[e_1,e_3] = e_3$ $[e_1,e_4] = e_4$ $[e_2,e_3] = -e_4$ $[e_2,e_4] = e_3$	$\begin{pmatrix} I_2 & 0_2 \\ \varphi & \begin{matrix} u & -v \\ v & u \end{matrix} \end{pmatrix}$ $\begin{pmatrix} I_2 & 0_2 \\ \varphi & \begin{matrix} -u & v \\ v & u \end{matrix} \end{pmatrix}$ $\begin{pmatrix} \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} & 0_2 \\ \varphi & \begin{matrix} u & -v \\ v & u \end{matrix} \end{pmatrix}$ $\begin{pmatrix} \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} & 0_2 \\ \varphi & \begin{matrix} -u & v \\ v & u \end{matrix} \end{pmatrix}$ where $\varphi = \begin{pmatrix} \varphi_1 & \varphi_2 \\ -\varphi_2 & \varphi_1 \end{pmatrix}$, and $u^2 + v^2 \neq 0$.

$\mathcal{G}$ Basis relations	Aut $(\mathcal{G})$
$\mathcal{G}_{4,3}$ $[e_1,e_2] = e_3 \quad [e_1,e_3] = e_4$	$\begin{pmatrix} a & 0 & 0 & 0 \\ c & d & 0 & 0 \\ r & s & ad & 0 \\ t & u & as & a^2d \end{pmatrix}$, $ad \neq 0$
$\mathcal{G}_{4,4}$ $[e_1,e_2] = e_3 \quad [e_1,e_4] = e_4$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a & b & 0 & 0 \\ r & s & b & 0 \\ t & 0 & 0 & u \end{pmatrix}$, $bu \neq 0$
$\mathcal{G}_{4,5}(\alpha,\beta)$ $[e_1,e_2] = e_2 \quad [e_1,e_3] = \alpha e_3$ $[e_1,e_4] = \beta e_4$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a & b & 0 & 0 \\ 0 & 0 & c & 0 \\ r & s & t & u \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 0 & b & 0 \\ 0 & c & 0 & 0 \\ r & s & t & u \end{pmatrix}$, <u>where</u> $t=0$ if $\beta=1, \alpha\neq 1$ $s=0$ if $\alpha=\beta\neq 1$ $s=t=0$ if $\alpha\neq\beta$ and $\beta\neq 1$
$\mathcal{G}_{4,6}(\alpha)$ $[e_1,e_3] = e_3 \quad [e_1,e_2] = e_2+e_3$ $[e_1,e_4] = \alpha e_4$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a & b & 0 & 0 \\ c & d & b & 0 \\ r & s & 0 & t \end{pmatrix}$, $bt \neq 0$
$\mathcal{G}_{4,7}$ $[e_1,e_2] = e_2+e_3$ $[e_1,e_3] = e_3+e_4 \quad [e_1,e_4] = e_4$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a & b & 0 & 0 \\ c & d & b & 0 \\ u & v & -d & b \end{pmatrix}$, $b \neq 0$
$\mathcal{G}_{4,8}(\alpha,\beta)$ $[e_1,e_2] = \alpha e_2 \quad [e_1,e_3] = \beta e_3-e_4$ $[e_1,e_4] = e_3+\beta e_4$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a & b & 0 & 0 \\ r & s & x & -y \\ t & u & y & x \end{pmatrix}$ <u>where</u> $s=u=0$ if $(\alpha-\beta)^2 \neq 1$ $s=u$ if $\beta-\alpha = 1$ $s=-u$ if $\beta-\alpha = -1$

Basis relations	Aut $(\mathcal{G})$
$\mathcal{G}_{4,9}(\alpha)$ $[e_2,e_3] = e_4 \quad [e_1,e_2] = (\alpha-1)e_2$ $[e_1,e_3] = e_3 \quad [e_1,e_4] = \alpha e_4$	$\underline{\alpha \neq 0,\ \alpha \neq 2}$: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ r & a & 0 & 0 \\ s & 0 & d & 0 \\ t & as & u & ad \end{pmatrix}$, $ad \neq 0$, $u = \frac{rd}{1-\alpha}$. $\underline{\alpha = 0}$: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ r & a & 0 & 0 \\ s & 0 & d & 0 \\ t & as & dr & ad \end{pmatrix}$, $ad \neq 0$, $\begin{pmatrix} -1 & 0 & 0 & 0 \\ r & 0 & b & 0 \\ s & c & 0 & 0 \\ t & -cr & -bs & -bc \end{pmatrix}$, $bc \neq 0$. $\underline{\alpha = 2}$: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ r & a & b & 0 \\ s & c & d & 0 \\ t & u & v & c_0 \end{pmatrix}$, $c_0 = ad - bc \neq 0$, $u = as - cr$, $v = bs - dr$.
$\mathcal{G}_{4,10}$ $[e_2,e_3] = e_4$ $[e_1,e_2] = e_2+e_3 \quad [e_1,e_3] = e_3$ $[e_1,e_4] = 2e_4$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ r & a & 0 & 0 \\ s & c & d & 0 \\ t & u & v & ad \end{pmatrix}$, $ad \neq 0$, $u = as - (c+d)r$, $v = -rd$.
$\mathcal{G}_{4,11}(\alpha)$ $[e_2,e_3] = e_4$ $[e_1,e_2] = \alpha e_2 - e_3$ $[e_1,e_3] = e_2+\alpha e_3 \quad [e_1,e_4] = 2\alpha e_4$	$\underline{\alpha \neq 0}$: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ r & a & b & 0 \\ s & -b & a & 0 \\ t & u & v & c_0 \end{pmatrix}$, $\alpha u + v = as + br$, $\alpha v - u = bs - ar$, $c_0 \neq 0$. $\underline{\alpha = 0}$: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ r & a & b & 0 \\ s & -b & a & 0 \\ t & u & v & c_0 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 & 0 & 0 \\ r & a & b & 0 \\ s & b & -a & 0 \\ t & u & v & c_0 \end{pmatrix}$, $u = -ra - sb$, $v = sa - rb$, $c_0 \neq 0$.

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